Baire measurable paradoxical decompositions

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Paradoxical decompositions

Suppose $\Gamma \curvearrowright^a X$ is an action of a group Γ on a space X. A **paradoxical decomposition** of a is a finite partition $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ of X and group elements $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \Gamma$ so that X is the disjoint union

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2. The free group on two generators $\mathbb{F}_2 = \langle a, b \rangle$ acts on itself via left translation. Let A_1, A_{-1}, B_1, B_{-1} be the words beginning with a, a^{-1}, b, b^{-1}, resp. This is almost a paradoxical decomposition (mod the identity) since $\mathbb{F}_2 = A_1 \sqcup aA_{-1} = B_1 \sqcup bB_{-1}$.

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3. The left translation action of $\mathbb{Z}$ on itself does not have a paradoxical decomposition.

How pathological are paradoxical decompositions?

Theorem (Dougherty-Foreman, 1994, answering Marczewski 1930)

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More generally, they showed every free Borel action of $\mathbb{F}_2$ on a Polish space by homeomorphisms has a paradoxical decomposition using pieces with the Baire property.

A generalization

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Theorem (M.-Unger)

Suppose a group acting on a Polish space by Borel automorphisms has a paradoxical decomposition. Then the action has a paradoxical decomposition where each piece has the Baire property.

Paradoxical decompositions and matchings

Suppose $\Gamma \curvearrowright^{a} X$ and $S \subseteq \Gamma$ is finite and symmetric. Let $G_{\mathrm{p}}(a, S)$ be the bipartite graph with vertex set $\{0, 1, 2\} \times X$ and

$$(i, x) G_{\mathrm{p}}(a, S)(j, y) \leftrightarrow ((\exists \gamma \in S) \gamma \cdot x = y) \wedge (i \neq j) \wedge (i = 0 \vee j = 0)$$

Claim

$G_p(a, S)$ has a perfect matching iff the action a has a paradoxical using group elements from S.

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Proof.

If $S = \{\gamma_1, \ldots, \gamma_n\}$, then given a perfect matching M, put $x \in A_i$ if $(0, x)$ is matched to $(1, \gamma_i \cdot x)$ and $x \in B_i$ if $(0, x)$ is matched to $(2, \gamma_i \cdot x)$. Then $\{A_1, \ldots, A_n, B_1, \ldots, B_n\}$ partitions the space, as do the sets $\gamma_i A_i$ and also $\gamma_i B_i$. □

Hall's theorem

Theorem (Hall)

A bipartite graph G with bipartition $\{B_0, B_1\}$ has a perfect matching iff for every finite subset F of B_0 or B_1,

$$|\mathrm{N}(F)| \geq |F|$$

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Laczkovich (1988) gave a negative answer to this question. Indeed, there is a Borel bipartite graph G where every vertex has degree 2, and there is no Borel perfect matching of G restricted to any comeager Borel set.

A version of Hall's theorem for Baire category

A weaker version of Hall's theorem is true for Baire category:

Theorem (M.-Unger)

Suppose G is a locally finite bipartite Borel graph on a Polish space with bipartition $\{B_0, B_1\}$ and there exists an $\epsilon > 0$ such that for every finite set F with $F \subseteq B_0$ or $F \subseteq B_1$,

$$|\mathrm{N}(F)| \geq (1 + \epsilon)|F|.$$

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Laczkovich's example shows that we cannot improve ϵ to 0.

How do we use Baire category?

Lemma

Suppose $f : \mathbb{N} \to \mathbb{N}$ *and* G *is a locally finite Borel graph on a Polish space* X*. Then there is a sequence* $\langle A_n \mid n \in \mathbb{N} \rangle$ *of Borel sets s.t.* $\bigcup_{n\in\mathbb{N}} A_n$ *is comeager and distinct* $x, y \in A_n$ *have* $d_G(x, y) > f(n)$*.*

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Proof.

Let $\langle U_i \mid i \in \mathbb{N} \rangle$ be a basis of open sets. Define sets $B_{i,r}$ by setting $x \in B_{i,r}$ if and only if $x \in U_i$ and for all $y \neq x$ in the closed r-ball around x, we have $y \notin U_i$.

Distinct $x, y \in B_{i,r}$ have $d_G(x, y) > r$. For fixed r, $X = \bigcup_i B_{i,r}$, since we can separate x from its r-ball by some U_i.

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Use the Baire category theorem to choose A_n to be some $B_{i,f(n)}$ that is nonmeager in the nth open set U_n. Hence $\bigcup_n A_n$ is comeager. □

Constructing perfect matchings

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Since G satisfies Hall's condition, it has a perfect matching. Take such a matching and remove an edge from it along with the two associated vertices. The resulting graph still has a perfect matching, so it satisfies Hall's condition. Repeat this process until we have constructed a matching of the entire graph. □

Constructing Baire measurable perfect matchings

Proof sketch of the Baire category version of Hall's theorem.

Take a Borel graph satisfying our strengthening of Hall's condition. Iteratively remove a Borel set of edges and their associated vertices of very large pairwise distance such that each edge individually comes from a matching of the graph. By our lemma, we can make the resulting set of removed edges comeager.

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We check that we preserve some form of Hall's condition after removing such a set of edges.

Small sets F will have $|\mathrm{N}(F)| \geq |F|$ since such a set can be near at most one edge we removed, and by construction each single edge we remove leaves a graph satisfying Hall's condition. Large sets F will still have $|\mathrm{N}(F)| \geq (1+\epsilon')|F|$ for some ϵ' since the number neighbors of F we have removed must be very small as a proportion of F. □

An amplification trick

Assume Γ acts by Borel automorphisms on a Polish space X and the action has some paradoxical decomposition using group elements from a finite symmetric set S. Hence, $G_{\mathrm{p}}(a, S)$ satisfies Hall's condition.

Claim

$G_p(a, S^2)$ satisfies the strengthened Hall condition $|\mathrm{N}(F)| \geq 2|F|$.

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Assume Γ acts by Borel automorphisms on a Polish space X and the action has some paradoxical decomposition using group elements from a finite symmetric set S. Hence, $G_{\mathrm{p}}(a, S)$ satisfies Hall's condition.

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Proof.

This is trivial for finite sets of the form $F = \{0\} \times F'$. For sets of the form $F = \{1, 2\} \times F'$, if we let $\{0\} \times F'' = \mathrm{N}_{G_{\mathrm{p}}(a,S)}(\{1, 2\} \times F')$, then $|F''| \geq |F|$, so

$$\begin{aligned} |\mathrm{N}_{G_{\mathrm{p}}(a,S^2)}(F)| &= |\mathrm{N}_{G_{\mathrm{p}}(a,S^2)}(\{1,2\} \times F')| \\ &\geq |\mathrm{N}_{G_{\mathrm{p}}(a,S)}(\{1,2\} \times F'')| \\ &\geq |\{1,2\} \times F''| \\ &\geq 2|F| \end{aligned}$$

This completes the proof

1. If $\Gamma \curvearrowright^{a} X$ is paradoxical, there is some S so $G_{\mathrm{p}}(a, S)$ has a perfect matching.
2. Since $G_{\mathrm{p}}(a, S)$ satisfies Hall's condition, $G_{\mathrm{p}}(a, S^2)$ satisfies the strengthened Hall condition where $|\mathrm{N}(F)| \geq 2|F|$
3. Apply our version of Hall's theorem to $G_{\mathrm{p}}(a, S^2)$ to find a Borel perfect matching on a comeager Borel set.
4. Use AC to extended this to a perfect matching on the whole graph $G_{\mathrm{p}}(a, S^2)$. The associated paradoxical decomposition of a will have pieces with the Baire property.

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(Note that this means our Baire measurable paradoxical decomposition will use more pieces than the original. This is known to be necessary by a result of Wehrung (1994), who showed any Baire measurable paradoxical decomposition of the 2-sphere needs at least 6 pieces, while there are decompositions from AC using 4).

A recent result proved using similar tools

Theorem (Grabowski, Máthé, Pikhurko, 2014)

Any two bounded subsets of $\mathbb{R}^3$ with nonempty interior and the same Lebesgue measure are equidecomposable using Lebesgue measurable pieces.

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They use the spectral gap in $\mathrm{SO}(3)$ to show they can satisfy the hypothesis of a theorem of Lyons and Nazarov on Lebesgue measurable matchings.

The von Neumann-Day problem

- A group Γ is **nonamenable** if the translation action of Γ on itself has a paradoxical composition. E.g. $\mathbb{F}_2$ is nonamenable.

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- The von Neumann-Day problem: is a group nonamenable iff it contains $\mathbb{F}_2$ as a subgroup?
- This problem was answered in the negative by Ol'shanskii (1980).
- Despite this negative answer, we would still like to show that in some sense, every paradoxical action inherits its paradoxicality from some related $\mathbb{F}_2$ action. (See for example the theorems of Whyte and Gaboriau-Lyons)

A Baire category solution

If a and b are actions of Γ and Δ on a space X, then the action b is said to be a-**Lipschitz**, if for every $\delta \in \Delta$, there is a finite set $S \subseteq \Gamma$ such that $\forall x \in X \exists \gamma \in S(\delta \cdot_b x = \gamma \cdot_a x)$.

Theorem

Suppose a is a nonamenable action of a group Γ on a Polish space X by Borel automorphisms. Then there is a free a-Lipschitz action of $\mathbb{F}_2$ on X by Baire measurable automorphisms.

The Ruziewicz problem

Theorem (Margulis-Sullivan (1980) $n \geq 4$ Drinfeld (1984) $n = 2, 3$)

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Using the work of Drinfeld-Margulis-Sullivan, this is equivalent to asking whether every Borel Lebesgue nullset is contained in a Borel Lebesgue nullset that has a Borel paradoxical decomposition.

A dichotomy?

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Prove a dichotomy theorem characterizing when a locally finite Borel graph has a Baire measurable perfect matching.

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Theorem (M.-Unger)

Suppose Γ is a finitely generated nonamenable group. Then the Cayley graph associated to any free Borel action of Γ has a Borel perfect matching on a comeager Borel set.