Thin–tall Boolean Algebras

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Finite–support iterations with symmetric systems as side conditions

Proper forcing is nice:

- Proper forcing notions preserve ω_1.
- Properness (due to Shelah) is preserved under countable support iterations.

Hence, granted the existence of a supercompact cardinal, one can build a model of PFA, the forcing axiom for proper forcings relative to collection of $\aleph_1$–many dense series (Baumgartner).

PFA has many consequences. One of them is $2^{\aleph_0} = \aleph_2$.

Problem: Force some consequence of PFA or, for that matter, something we can force by iterating proper forcing, together with $2^{\aleph_0} > \aleph_2$.

Countable support iterations won't do. In fact, at stages of uncountable cofinality we are adding generics, over all previous models, for $\mathrm{Add}(1, \omega_1)$ (= adding a Cohen subset of ω_1); in particular we are collapsing the continuum of all those previous models to $\aleph_1$. Hence, in the final model necessarily $2^{\aleph_0} \leq \aleph_2$.

Bigger support won't work either: The preservation lemma for properness doesn't work in this context.

Finite–support iterations won't work either; in fact, any finite–support iteration of non–c.c.c. forcings collapses ω_1.

A solution: Use finite supports, together with countable elementary substructures of some $H(\theta)$ as side conditions affecting the whole iteration or initial segments of the iteration in order to ensure properness (the idea of using countable structures as side conditions in order to "force" a non–proper forcing to become proper is old (Todorčević, 1980's, implicit in work of Baumgartner (adding a club of ω_1 by finite conditions)), but this was not done in the context of actual iterations).

Typically we will want our iteration to have the $\aleph_2$–c.c. (after all we are interested in $2^{\aleph_0}$ arbitrarily large).The natural approach of using finite $\in$–chains of structures won't work, though, since we have too many structures and would therefore lose the $\aleph_2$–c.c. We will replace $\in$–chains of structures by "matrices" of structures with suitable symmetry properties. If we start with CH and consider only iterands with the $\aleph_2$–c.c., we might succeed.

Symmetric systems of elementary substructures

Given a set N, δ_N will denote $N \cap \omega_1$ (the height of N).

Definition

Let θ be a cardinal and $T \subseteq H(\theta)$ (such that $\bigcup T = H(\theta)$). A finite set $\mathcal{N} \subseteq [H(\theta)]^{\aleph_0}$ is a T*–symmetric system* iff the following holds for all N, N_0, $N_1 \in \mathcal{N}$:

(1) $(N; \in, Y) \preccurlyeq (H(\theta); \in, T)$

(2) If $\delta_{N_0} = \delta_{N_1}$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0; \in, T) \longrightarrow (N_1; \in, T)$$

Furthermore, Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) If $\delta_{N_0} = \delta_{N_1}$ and $N \in N_0 \cap \mathcal{N}$, then $\Psi_{N_0,N_1}(N) \in \mathcal{N}$.

(4) If $\delta_{N_0} < \delta_{N_1}$, then there is some $N_1' \in \mathcal{N}$ such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.

- Symmetric systems had previously been considered in (at least) work of Todorčević, Abraham–Cummings and Koszmider. Again, not in the context of forcing iterations.
- The def. of symmetric system guarantees that

  (4)' if $N_0, N_1 \in \mathcal{N}$ and $\delta_{N_0} < \delta_{N_1}$, then there is some $N_0' \in N_1 \cap \mathcal{N}$ such that $\delta_{N_0'} = \delta_{N_0}$ and $N_0 \cap N_1 = N_0 \cap N_0'$.

(In fact, $N_0' = \Psi_{N_1', N_1}(N_0)$, where $N_1' \in \mathcal{N}$ is such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.) This property is important in many applications. Sometimes it is enough to keep (1)–(3) and weaken (4) to (4)'. The resulting object is called *partial T–symmetric system*.

Two amalgamation lemmas

1st amalgamation lemma: If $\mathcal{N}$ and $\mathcal{N}'$ are T–symmetric systems, $(\bigcup \mathcal{N}) \cap (\bigcup \mathcal{N}') = X$, and there are enumerations $(N_i)_{i<n}$ and $(N_i')_{i<n}$ of $\mathcal{N}$, $\mathcal{N}'$, resp., for which there is an isomorphism

$$\Psi : (\bigcup \mathcal{N}; \in, N_i, T, X) \longrightarrow (\bigcup \mathcal{N}'; \in, N_i', T, X)$$

then $\mathcal{N} \cup \mathcal{N}'$ is a T–symmetric system.

2nd amalgamation lemma: Let $\mathcal{N}$ be a T–symmetric system and $M \in \mathcal{N}$. Suppose $\mathcal{M} \in M$ is a T–symmetric system such that $\mathcal{N} \cap M \subseteq \mathcal{M}$. Let

$$\mathcal{N}^M(\mathcal{M}) = \mathcal{N} \cup \{\Psi_{M,M'}(N) \ : \ N \in \mathcal{M},\ M' \in \mathcal{N} \ : \ \delta_{M'} = \delta_M\}$$

Then $\mathcal{N}^M(\mathcal{M})$ is a T–symmetric system.

Corollaries

Let

$$\mathrm{Symm}_T = (\{\mathcal{N} : \mathcal{N}\ T\text{–symmetric system}\}, \supseteq)$$

Using 1st amalgamation lemma:

Corollary 1 (CH) Symm_T is $\aleph_2$–Knaster.

Corollary 2 (CH) Symm_T adds new reals but preserves CH.

Using 2nd amalgamation lemma:

Corollary 3 Symm_T is proper.

Iterating: General template of the constructions.

Start with CH, let κ regular with $2^{<\kappa} = \kappa$. Fix suitable $T \subseteq H(\kappa)$. Let $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ be such that for all α, a condition in $\mathcal{P}_\alpha$ is a pair $q = (F, \Delta)$ such that:

(1) F is a finite function such that $\mathrm{dom}(F) \subseteq \alpha$ ($\mathrm{dom}(F)$ is the support of q).

(2) Δ is a finite set of pairs (N, γ), where $N \in [H(\kappa)]^{\aleph_0}$, $\gamma \leq \alpha$, $\gamma \leq \sup(N \cap \kappa)$, and where $\mathrm{dom}(\Delta)$ is a (partial) T–symmetric system (γ is the *marker* associated to N).

(3) For all $\beta < \alpha$,

$$q|_\beta := (F \restriction \beta, \{(N, \min\{\gamma, \beta\}) : (N, \gamma) \in \Delta\})$$

is a $\mathcal{P}_\beta$–condition.

(4) For every $\xi \in \mathrm{dom}(F)$,

$$q|_\xi \Vdash_{\mathcal{P}_\xi} F(\xi) \in \Phi^*(\xi)$$

where $\Phi^*(\xi)$ is a $\mathcal{P}_\xi$–name for a suitable forcing, and $\Phi^*(\xi) = \Phi(\xi)$ if $\Phi(\xi)$ is a $\mathcal{P}_\xi$–name for a suitable forcing (and where Φ is a suitable bookkeeping function on κ).

(5) For every $\xi \in \mathrm{dom}(F)$ and every $(N, \gamma) \in \Delta$, if $\xi \leq \gamma$ and $\xi \in N$, then

$$q|_\xi \Vdash_{\mathcal{P}_\xi} F(\xi) \text{ is } (N[\dot{G}_\xi], \Phi^*(\xi))\text{–generic}$$

Given $\mathcal{P}_\alpha$–conditions $q_0 = (F_0, \Delta_0)$, $q_1 = (F_1, \Delta_1)$, $q_1 \leq_\alpha q_0$ iff

(a) for every $(N, \gamma) \in \Delta_0$ there is some $\gamma' \geq \gamma$ such that $(N, \gamma') \in \Delta_1$,

(b) $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$, and

(c) for every $\xi \in \mathrm{dom}(F_0)$,

$$q_0|_\xi \Vdash_{\mathcal{P}_\xi} F_1(\xi) \leq_{\Phi^*(\xi)} F_0(\xi)$$

Typical properties

(1) $\mathcal{P}_\beta$ is always a complete suborder of $\mathcal{P}_\alpha$ whenever $\beta < \alpha$: Thanks to the markers γ in $(N, \gamma) \in \Delta$.

(2) Each $\mathcal{P}_\beta$ is typically $\aleph_2$–c.c.: This often uses CH and standard Δ–system arguments as in the proof of Corollary 1.

(3) Properness of $\mathcal{P}_\alpha$: We define a sequence $(\mathcal{M}_\alpha)_{\alpha \leq \kappa}$ of clubs of $[H(\kappa)]^{\aleph_0}$ (of increasing "richness"); e.g., by picking increasing sequence $(\theta_\alpha)_{\alpha \leq \kappa}$ of cardinals above κ and letting

$$\mathcal{M}_\alpha = \{N^* \cap H(\kappa) \,:\, N^* \preccurlyeq H(\theta_\alpha), T, \Phi, (\theta_\beta)_{\beta < \alpha} \in N^*\}$$

Proof by induction on α: Let $p \in \mathcal{P}_\alpha \cap N^*$, $N^* \preccurlyeq H(\theta)$ countable containing everything relevant. We build q^* from q by essentially adding $(N, \min\{\alpha, \sup(N \cap \kappa)\})$ to Δ_p. We argue that q^* can be built and is $(N^*, \mathcal{P}_\alpha)$–generic. For this, let $A \in N^*$, $A \subseteq \mathcal{P}_\alpha$ maximal antichain, and let $q \in \mathcal{P}_\alpha$ extend both a condi-
tion $t \in A$ and q^*. We want to find $r \in N \cap A$, r compatible with q.

Case $\alpha = 0$: Corollary 3.

Case $\alpha = \beta + 1$: Usually easy, since, by definition, $q|_\beta \Vdash_{\mathcal{P}_\beta} F_q(\beta)$ is $N[\dot{G}_\beta], \Phi^*(\beta))$–generic (if $\beta \in \mathrm{dom}(F_q)$).

Case $\alpha \neq 0$ **limit**: The case when $\mathrm{cf}(\alpha) \neq \omega_1$ is typically easy, since then there is $\sigma \in \sigma \in N \cap \alpha$ bounding the support of (some condition in A extended by) q (obvious when $\mathrm{cf}(\alpha) = \omega$, using that $|A| \leq \aleph_1$ if $\mathrm{cf}(\alpha) \geq \omega_2$). Then we apply induction hypothesis to σ: Working in $N[\dot{G}_\sigma]$, find $r \in A$ such that $r|_\sigma \in \dot{G}_\sigma$, r compatible with everything $N[\dot{G}_\sigma]$ can see, $\mathrm{dom}(F_r) \cap [\sigma, \alpha) = \emptyset$. By extending $q|_\sigma$ we can assume $r \in N$ (since, by induction hypothesis, $q|_\sigma$ is $(N[\dot{G}_\sigma], \mathcal{P}_\sigma)$–generic). Now we can amalgamate q and r into a condition.

In the case $\mathrm{cf}(\alpha) = \omega_1$, go to the blackboard.

A typical application:

Theorem

(A.–Mota, A generalization of Martin's Axiom, Israel J. Math., to appear) (GCH) *For every regular* $\kappa \geq \omega_2$*, there is a proper* $\aleph_2$*–c.c. forcing notion forcing* $MA^{1.5}_{<\kappa} + 2^{\aleph_0} = \kappa$*.*

Here $\mathrm{MA}^{1.5}_{\lambda}$ is the forcing axiom for the class of $\aleph_{1.5}$–c.c. partial orders relative to collections of $\leq \lambda$–many dense sets, where $\mathbb{P}$ has the $\aleph_{1.5}$–c.c. iff for every $\theta > |TC(\mathbb{P})|$ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for any finitely many $N_0, \ldots, N_m \in D$ and every $p \in \mathbb{P}$, if $p \in N_i$ for all i with δ_{N_i} minimal among $N_0, \ldots, N_m$, then there is $q \leq_{\mathbb{P}} p$ such that q is $(N_k, \mathbb{P})$–generic for all $k \leq m$.

$\mathrm{MA}^{1.5}_{\lambda}$ of course extends MA_{λ} but also has, for example, many consequences at the level of strong failures of Club Guessing at ω_1.

Some nice spin–offs:

For every κ, there is a homogeneous $\aleph_2$–c.c. proper forcing $\text{Add}_{\mathbb{B}}(\kappa)$ adding κ–many Baumgartner clubs to ω_1 (CH not needed!).

$$\frac{\text{Add}_{\mathbb{B}}(\kappa)}{\text{Adding a Baumgartner club}} = \frac{\text{Add}(\kappa, \omega)}{\text{Cohen forcing}}$$

In particular, $\text{Add}_{\mathbb{B}}(\kappa)$ has applications in the context of cardinal characteristics for ${}^{\omega_1}\omega_1$ and $[\omega_1]^{\aleph_1}$.
$\text{Add}_{\mathbb{B}}(\kappa)$ also figures prominently in the construction, in ZFC, of a forcing notion collapsing $\aleph_3$ but preserving all other cadinals (the existence of this forcing answers a 1983 question of Abraham, who built in ZFC a forcing collapsing $\aleph_2$ and preserving all other cardinals).

Changing the side conditions (I): Larger structures.

The theory of T–symmetric systems goes through unchanged if we re-
place $|N| = \aleph_0$ with $|N| = \lambda$ (and we can also ask that ${}^{<\lambda}N \subseteq N$).

We would then expect to be able to iterate $<\lambda$–closed forcings which are suitably λ–proper (i.e., proper with respect to sufficiently many elem. substructures N such that $|N| = \lambda$, perhaps in the strong sense of the def. of $\aleph_{1.5}$–c.c.).

This doesn't work: In the above inductive proof of properness it is essential that supports be finite. In fact, this cannot work. Otherwise we should be able to build models falsifying instances of Club–Guessing on ω_2 which hold true in ZFC! Specifically we would be able to destroy the Club–Guessing in the following slide.

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Theorem

(Shelah, Claim 3.3 in Colouring and non-productivity of $\aleph_2$–c.c., Ann. Pure and Applied Logic, vol. 84, 2 (1997), 153–174)

Let $\kappa > \omega_1$ be a regular cardinal. Then for every stationary $S \subseteq S^{\kappa^+}_{\kappa}$ there is a club–sequence $\langle C_\delta : \delta \in S \rangle$ such that for all $\delta \in S$,

- $\mathrm{ot}(C_\delta) = \kappa$, *and*
- $\mathrm{cf}(C_\delta(\alpha + 1)) = \kappa$ *for all* $\alpha < \kappa$,

and such that for every club $D \subseteq \kappa^+$ there is some $\delta \in S$ (equivalently, stationary many $\delta \in S$) such that

$$\{\alpha < \kappa : C_\delta(\alpha + 1) \in D\}$$

is stationary (where $C_\delta(\beta)$ is the β–th member of C_δ).

See also D. Soukup and L. Soukup, *Club guessing for dummies* for a nicely written proof of the above.

Question (Shelah, Question 5.4 in *On what I do not understand (and have something to say): Part I*, Fundamenta Math., vol. 166, 1–2 (2000), 1–82.)

Is it true in ZFC that for every regular cardinal $\kappa \geq \omega_1$ there is a club–sequence $\vec{C} = \langle C_\delta : \delta \in S^{\kappa^+}_\kappa \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ such that for every club $D \subseteq \kappa^+$ there is some δ such that

$$\{\alpha < \kappa : \{C_\delta(\alpha + 1), C_\delta(\alpha + 2)\} \subseteq D\}$$

is stationary?

According to Shelah in the above paper, if there is a club–sequence as in the above question on $S^{\kappa^+}_{\kappa}$ and GCH holds, then there is a κ^+–Souslin tree. In particular, an affirmative answer to above question would yield an affirmative answer to the following well–known open question.

Question: Does GCH imply that there is an ω_2–Souslin tree?

The following (from A., *The consistency of a club–guessing failure at the successor of a regular cardinal*, in "Infinity, computability, and metamathematics: Festschrift celebrating the 60th birthdays of Peter Koepke and Philip Welch," 2014, 5–27) answers the above question:

Theorem

(GCH) For every regular cardinal $\kappa \geq \omega_1$ there is a cardinal–preserving poset forcing that there is no club–sequence $\vec{C} = \langle C_\delta \,:\, \delta \in S^{\kappa^+}_\kappa \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ and such that for every club $D \subseteq \kappa^+$ there is no δ such that

$$\{\alpha < \kappa \,:\, \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$

is stationary.

Proof by a $<\kappa$–support iteration of length κ^{++} using, as side conditions, symmetric systems of size $<\kappa$ of models N such that $|N| = \kappa$ and ${}^{<\kappa}N \subseteq N$. Proof of κ–properness is direct, not by induction.

Also using this type of side conditions:

Theorem

(Mota–Weiss) (GCH) For every regular κ there is a cardinal–preserving poset forcing the existence of a superatomic Boolean algebra of width κ and height κ^{++}.

Baumgartner–Shelah (1987) did the case $\kappa = \omega$; it was open whether $\kappa > \omega$ is possible.

Changing the side conditions (II): Structures of two types

Neeman considers side conditions consisting of structures of two cardinalities types (small and large) $\{Q_0, \ldots, Q_m\}$ such that for all i,

1. $Q_i \in Q_{i+1}$
2. If Q_i is large and Q_{i+1} is small, then there is some $j < i$ such that $Q_i \cap Q_{i+1} = Q_j$.

In his (main) applications:

- small = countable, large = transitive (some $H(\lambda)$). Uses these to build a model of PFA using finite supports.
- small = countable, large = cardinality $\aleph_1$. Uses these to add objects on ω_2 (various types of $\square_{\omega_1}$–sequences).

Now we want to solve

$$\frac{?}{\text{Lin. ordered side conds. 2 types}} = \frac{\text{Symmetric systems}}{\in\text{–chains structures same card.}}$$

Definition Let θ be a cardinal and $T \subseteq H(\theta)$ (such that $\bigcup T = H(\theta)$). Let κ be an infinite cardinal. [Given a set N, δ_N will now denote $\sup(N \cap \kappa^{++})$.] A set $\mathcal{N} \subseteq H(\theta)$ with $|\mathcal{N}| < \kappa$ is a *T–symmetric system of type $\{\kappa, \kappa^+\}$* iff the following holds for all Q_0, Q_1, $Q \in \mathcal{N}$:

(1) $(Q; \in, T) \preccurlyeq (H(\theta); \in, T)$, $|Q| \in \{\kappa, \kappa^+\}$, and ${}^{<|Q|}Q \subseteq Q$.

(2) If $\delta_{Q_0} = \delta_{Q_1}$, then the following holds.

   (a) $Q_0 \cap \kappa^{++} = Q_1 \cap \kappa^{++}$

   (b) There is a (unique) isomorphism

$$\Psi_{Q_0,Q_1} : (Q_0; \in, T) \longrightarrow (Q_1; \in, T)$$

   Furthermore, Ψ_{Q_0,Q_1} is the identity on $Q_0 \cap Q_1$.

(3) If $\delta_{Q_0} = \delta_{Q_1}$ and $Q \in Q_0$, then $\Psi_{Q_0,Q_1}(Q) \in \mathcal{N}$.

(4) If $\delta_{Q_0} < \delta_{Q_1}$, then there is some $Q_1' \in \mathcal{N}$ such that $\delta_{Q_1'} = \delta_{Q_1}$ and such that the following holds.

   (a) If $Q_0 \in Q_1'$, then $Q_0 \cap Q_1' \in \mathcal{N}$.
   (b) If $Q_0 \notin Q_1'$, then there is some $N \in Q_1' \cap \mathcal{N}$ such that $|N| = \kappa^+$, $Q_0 \in N$, $Q_1' \cap N \in \mathcal{N}$ and $\delta_{Q_1' \cap N} < \delta_{N_0}$.

Two amalgamation lemmas

1st amalgamation lemma: If $\mathcal{N}$ and $\mathcal{N}'$ are T–symmetric systems of type $\{\kappa, \kappa^+\}$, $(\bigcup \mathcal{N}) \cap (\bigcup \mathcal{N}') = X$, and there are enumerations $(N_i)_{i<n}$ and $(N'_i)_{i<n}$ of $\mathcal{N}$, $\mathcal{N}'$, resp., for which there is an isomorphism

$$\Psi : (\bigcup \mathcal{N}; \in, N_i, T, X) \longrightarrow (\bigcup \mathcal{N}'; \in, N'_i, T, X)$$

then $\mathcal{N} \cup \mathcal{N}'$ is a T–symmetric system of type $\{\kappa, \kappa^+\}$.

2nd amalgamation lemma: Let $\mathcal{N}$ be a T–symmetric system and $M \in \mathcal{N}$ of type $\{\kappa, \kappa^+\}$. Suppose $\mathcal{M} \in M$ is a T–symmetric system of type $\{\kappa, \kappa^+\}$ such that $\mathcal{N} \cap M \subseteq \mathcal{M}$. Let

$$\mathcal{N}^M(\mathcal{M}) = \mathcal{N} \cup \{\Psi_{M,M'}(N) \,:\, N \in \mathcal{M},\, M' \in \mathcal{N} \,:\, \delta_{M'} = \delta_M\}$$

Then $\mathcal{N}^M(\mathcal{M})$ is a T–symmetric system of type $\{\kappa, \kappa^+\}$.

Corollaries

Let

$$\text{Symm}_T = (\{\mathcal{N} : \mathcal{N}\ T\text{–symmetric system of type } \{\kappa, \kappa^+\}\}, \supseteq)$$

Symm_T is clearly $<\kappa$–closed.

Using 1st amalgamation lemma:

Corollary 1 ($2^{\kappa^+} = \kappa^{++}$) Symm_T is κ^{+++}–Knaster.

Corollary 2

- ($2^{\kappa^+} = \kappa^{++}$) Symm_T adds new subsets of κ^+ but preserves $2^{\kappa^+} = \kappa^{++}$.
- ($2^{\kappa} = \kappa^+$) Symm_T adds new subsets of κ but preserves $2^{\kappa} = \kappa^+$.

Using 2nd amalgamation lemma:

Corollary 3 Symm_T is proper for structures Q with $|Q| \in \{\kappa, \kappa^+\}$ such that ${}^{<|Q|}Q \subseteq Q$.

Can we hope to iterate suitably proper and $\aleph_1$–proper forcing, with finite supports, using symmetric systems of type $\{\aleph_0, \aleph_1\}$ with markers to ensure preservation of ω_1 and ω_2? (Remember that having to rely on infinite supports was a problem for proving properness in our context.)

No: Go to the board.

In fact, if there were a reasonable iteration theory here, starting from GCH we would be able to add clubs of ω_2 by finite conditions in length, say, ω_3, and in the end we would have killed Club–Guessing on ω_2 (which is a ZFC–theorem)!

An application of symmetric systems of type $\{\aleph_0, \aleph_1\}$: Adding a sBa of width ω and height ω_3

The following question remained open since the work of Baumgartner–Shelah 1987:

Question

Is the existence of a superatomic Boolean algebra of width ω and height ω_3 consistent?

Theorem

(about a week ago, 85% true) (GCH) There is a proper, $\aleph_1$–proper, and $\aleph_3$–Knaster forcing notion adding a superatomic Boolean algebra of width ω and height ω_3.

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Given a Boolean algebra $\mathbb{B}$ let $\mathcal{I}(\mathbb{B})$ be the ideal of $\mathbb{B}$ generated by its atoms. For an ordinal α, define the α–th Cantor–Bendixson ideal $\mathcal{J}^{\alpha}(\mathbb{B})$ on $\mathbb{B}$:

- $\mathcal{J}^{0}(\mathbb{B}) = \{0\}$
- letting $\pi_{\alpha} : \mathbb{B} \longrightarrow \mathbb{B}/\mathcal{J}^{\alpha}(\mathbb{B})$ the canonical projection, $\mathcal{J}^{\alpha+1}(\mathbb{B}) = \pi_{\alpha}^{-1}(\mathcal{I}(\mathbb{B}/\mathcal{I}^{\alpha}(\mathbb{B})))$.
- If $\alpha \neq 0$ is limit, $\mathcal{J}^{\alpha}(\mathbb{B}) = \bigcup_{\beta<\alpha} \mathcal{J}^{\beta}(\mathbb{B})$.

$\mathbb{B}$ is superatomic iff there is some α such that $\mathbb{B} = \mathcal{J}^{\alpha+1}(\mathbb{B})$. The least such α is the height of $\mathbb{B}$ (denoted $\mathrm{ht}(\mathbb{B})$). $\mathbb{B}$ has width κ iff $|\mathbb{B}/\mathcal{J}^{\beta}(\mathbb{B})| = \kappa$ for every $\beta < \mathrm{ht}(\mathbb{B})$.

Definition

(essentially due to Baumgartner) Let κ, λ be infinite cardinals. An LCS($\kappa \times \lambda$)–*structure* is a pair $(\leq, b)$ where

- $\leq$ is a partial order on $\kappa \times \lambda$.
- If $(\alpha, \beta) < (\alpha', \beta')$, then $\beta < \beta'$.
- For every $(\alpha, \beta) \in \kappa \times \lambda$ and every $\delta < \beta$, $\{\gamma : (\gamma, \delta) < (\alpha, \beta)\}$ is infinite.
- $b : [\kappa \times \lambda]^2 \longrightarrow [\kappa \times \lambda]^{<\omega}$
- For all ν_0, $\nu_1 \in \kappa \times \lambda$,
  - for all $\nu \in b(\{\nu_0, \nu_1\})$, $\nu \leq \nu_0$ and $\nu \leq \nu_1$, and
  - for every ν, if $\nu \leq \nu_0$ and $\nu \leq \nu_1$, then there is some $\nu' \in b(\{\nu_0, \nu_1\})$ such that $\nu \leq \nu'$.

  (b is called *a barrier function for* $\leq$).

Proposition

(Baumgartner) Let κ, λ be infinite cardinals. If there is an LCS($\kappa \times \lambda$)*–structure, then there is a superatomic Boolean algebra of width κ and height λ.*

Proof of theorem: We build a proper $\aleph_1$–proper $\aleph_3$–Knaster forcing $\mathcal{P}$ adding an $\mathrm{LCS}(\omega \times \omega_3)$–stucture as follows. Let $T \subseteq H(\omega_3)$ code $(e_\beta)_{\beta<\omega_3}$ where $e_\beta : |\beta| \longrightarrow \beta$ bijection for all β.

Conditions are quadruples $(\mathcal{N}, \mathcal{A}, \leq, b)$, where:

(1) $\mathcal{N}$ is a T–symmetric system of type $\{\aleph_0, \aleph_1\}$.

(2) $\mathcal{A} \subseteq \mathcal{N}$

(3) $\leq$ is a partial order such that $\mathrm{dom}(\leq) \subseteq \omega \times \omega_3$ and $| \leq | < \aleph_0$.

(4) For all $(\alpha, \beta), (\alpha', \beta') \in \mathrm{dom}(\leq)$, if $(\alpha, \beta) \leq (\alpha', \beta')$ and $(\alpha, \beta) \neq (\alpha', \beta')$, then $\beta \in \beta'$.

(5) $b : [\mathrm{dom}(\leq)]^2 \longrightarrow [\mathrm{dom}(\leq)]^{<\omega}$ is a barrier function for $\leq$.

(6) For all $\nu_0, \nu_1 \in \mathrm{dom}(\leq)$ and all $Q \in \mathcal{A}$, if $\{\nu_0, \nu_1\} \in Q$, then $b(\{\nu_0, \nu_1\}) \in Q$.

(7) For all $\nu_0, \nu_1 \in \mathrm{dom}(\leq)$ and all $M \in \mathcal{A}$, if $|M| = \aleph_0$ and $\{\nu_0, \nu_1\} \subseteq \bigcup\{X \in M \,:\, |X| = \aleph_1\}$, then $b(\{\nu_0, \nu_1\}) \subseteq \bigcup\{X \in M \,:\, |X| = \aleph_1\}$.

(8) For all $\nu_0, \nu_1 \in \mathrm{dom}(\leq)$, $M \in \mathcal{A}$ and $N \in \mathcal{N} \cap M$, if $|M| = \aleph_0$, $|N| = \aleph_1$, $\nu_0 \in M$, $\nu_1 \in N$, and $\nu_0 \leq \nu_1$, then there is some $\nu \in M \cap N$ such that $\nu_0 \leq \nu \leq \nu_1$.

Given $\mathcal{P}$–conditions $p_0 = (\mathcal{N}_0, \mathcal{A}_0, \leq_0, b_0)$,
$p_1 = (\mathcal{N}_1, \mathcal{A}_1, \leq_1, b_1)$, we say that p_1 extends p_0 iff

- $\mathcal{N}_0 \subseteq \mathcal{N}_1$,
- $\mathcal{A}_0 \subseteq \mathcal{A}_1$,
- $\mathrm{dom}(\leq_0) \subseteq \mathrm{dom}(\leq_1)$ and $\leq_1 | \mathrm{dom}(\leq_0) = \leq_0$, and
- $b_1 \restriction [\mathrm{dom}(\leq_0)]^2 = b_0$.

Conjecture: One can force a sBa of width κ and height κ^{+++} for any given regular $\kappa \geq \omega_1$.

Should be doable combining these ideas with the ideas of Mota–Weiss one con prove the following.

Question: Can
one force existence of sBa of width κ and height λ for $\lambda > \kappa^{+++}$?

Conjecture: The answer should again be yes, at least for $\lambda = \kappa^{+4}$, using symmetric systems of Neeman's side conditions of three types.