Universally Baire subsets of 2^κ

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Joint work with Matteo Viale

26 September 2015

From now on...

We work in ZFC unless clearly specified.

Today

We will develop a basic theory of universally Baire subsets of 2^κ.

Motivation 1; generic absoluteness

Theorem (Woodin)

Suppose there are proper class many Woodin cardinals. Then for any poset P,

$$(\mathrm{L}(\mathbb{R}), \in; \mathbb{R})^{V} \equiv (\mathrm{L}(\mathbb{R}), \in; \mathbb{R})^{V^{P}}.$$

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Theorem (Viale)

Suppose that MM^{+++} holds and that there are proper class many Σ_2 reflecting cardinals which are limits of supercompacts. Then for any poset P which is stationary set preserving and which preserves MM^{+++},

$$\big(\mathrm{L}(\mathrm{Ord}^{\omega_1}), \in; \mathcal{P}(\omega_1)\big)^V \equiv \big(\mathrm{L}(\mathrm{Ord}^{\omega_1}), \in; \mathcal{P}(\omega_1)\big)^{V^P}.$$

Motivation; iterable structures and generic absoluteness

Theorem (Woodin)

Suppose that there is a countable transitive model of ZFC with ω-many Woodin cardinals which is iterable. Then for any poset P,

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Note: The existence of such an iterable structure follows from strong axioms of infinity such as the existence of proper class many Woodins, and PFA.

Iterable structures and beyond generic absoluteness

Theorem (Woodin)

Suppose that M and N are models of ZFC satisfying "$\mathsf{M}_\omega^\#$ exists and it is iterable". Assume that

$$(\omega, <)^M = (\omega, <)^N \text{ and } (\mathsf{M}_\omega^\#)^M = (\mathsf{M}_\omega^\#)^N.$$

Then

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Note:

1. $\mathsf{M}_\omega^\#$ is a *natural* example of an iterable structure with ω-many Woodin cardinals.
2. The existence of $\mathsf{M}_\omega^\#$ and its iterability follows from strong axioms of infinity such as the existence of proper class many Woodins, and PFA.

The Conjecture

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Suppose that M is a model of ZFC + MM^{+++} + "the generic nice UBH" with a supercompact cardinal which is a limit of supercompacts, and that N is a model of ZFC. Assume that

1. $(\omega_1, <)^M = (\omega_1, <)^N$, and
2. for any ω_1-sequence $(A_\alpha \mid \alpha < \omega_1)$ in M consisting of universally sets of reals in M, there is a corresponding sequence $(B_\alpha \mid \alpha < \omega_1)$ in N which are universally Baire sets of reals in N such that

$$\left(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1}, (A_\alpha \mid \alpha < \omega_1)\right)^M \prec_{\boldsymbol{\Sigma}_2} \left(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1}, (B_\alpha \mid \alpha < \omega_1)\right)^N.$$

Then

$$\left(\mathrm{L}(\mathcal{P}(\omega_1)), \in; \mathcal{P}(\omega_1)\right)^M \equiv \left(\mathrm{L}(\mathcal{P}(\omega_1)), \in; \mathcal{P}(\omega_1)\right)^N.$$

Key tool; iterable structures and extender algebras

Recall (Woodin)

Suppose that there is a countable transitive model of ZFC with ω-many Woodin cardinals which is iterable. Then for any poset P,

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What is behind is as follows:

Theorem (Woodin)

Let M be a set-sized transitive model of ZFC with a Woodin cardinal δ which is iterable. Then there is a complete Boolean algebra B_δ in M such that for any set x in V, there is an elementary embedding $j \colon M \to N$ and a $j(B_\delta)$-generic filter g over N in V such that x is in $N[g]$.

Such a B_δ is called an extender algebra.

From iterable structures to generic absoluteness

Points

1. Absoluteness of iterability of M between V and V^P is essential for generic absoluteness between V and V^P.

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2. Iterability of M is witnessed by a mathematical object called an iteration strategy Σ.
   1. When M is countable, Σ can be coded by a set of reals and this set of reals is usually universally Baire set of reals.
   2. When M is of size ω_1, Σ can be coded by a subset of $\mathcal{P}(\omega_1)$.
      Note: For Conjecture, we need M to be of size ω_1.

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3. For $\mathrm{L}(\mathbb{R})$ generic absoluteness result by Woodin, the countability of M is essential to ensure the existence of B_δ-generic filters over M in V (Baire Category Theorem).

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4. For Conjecture, we would need the existence of B_δ-generic filters over M in V when M is of size ω_1.
   This is where forcing axioms come in.

Point 1 & 2; iteration strategies and universally Baire sets

Theorem (Martin and Steel; Woodin)

Suppose that there are proper class many Woodin cardinals and that the generic nice UBH holds. Then for any countable $Z \prec_{\mathbf{\Sigma}_{2015}} V$, if M is the transitive collapse of Z, then there is an iteration strategy Σ for M such that Σ can be coded by a universally Baire set of reals.

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Theorem (Viale, I.)

Suppose that there are proper class many Woodin cardinals and that the generic nice UBH holds. Then for any $Z \prec_{\mathbf{\Sigma}_{2015}} V$ of size ω_1 with $\omega_1 \subseteq Z$, if M is the transitive collapse of Z, then there is an iteration strategy Σ for M such that

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1. there is an ω_1-sequence $(A_\alpha \mid \alpha < \omega_1)$ of universally Baire sets of reals such that Σ is $\mathbf{\Pi}_2$-definable over $\left(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1}, (A_\alpha \mid \alpha < \omega_1)\right)$, and

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2. Σ can be coded by a universally Baire subset of $\mathcal{P}(\omega_1)$.

Note: The second condition in Conjecture is motivated by ①.

Point 4; forcing axioms and generic filters over a structure of size ω_1

Observation

Suppose that MM^{++} holds. Let δ be a Woodin cardinal. Then there is a $Z \prec_{\mathbf{\Sigma}_{2015}} V$ of size ω_1 with $\omega_1 \subseteq Z$ and $\delta \in Z$ such that if $\pi \colon Z \to M$ is the collapsing map, then there is a $\pi(B_\delta)$-generic filter g over M in V such that

$$(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1})^{M[g]} \prec_{\mathbf{\Sigma}_0} (H_{\omega_2}, \in, \mathrm{NS}_{\omega_1})^{V}.$$

Motivation 2; generic absoluteness and universally Baireness

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Many generic absoluteness results obtained from large cardinals can be extracted as the universally Baireness of specific sets of reals.

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Typical example:

Theorem (Steel; Woodin)

The following are equivalent:

1. For any poset P,
$$(\mathrm{L}(\mathbb{R}), \in; \mathbb{R})^V \equiv (\mathrm{L}(\mathbb{R}), \in; \mathbb{R})^{V^P}.$$
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We would like to extract Viale's generic absoluteness result in such a way that certain subset of $\mathcal{P}(\omega_1)$ (maybe a "sharp" for $\mathrm{L}(\mathrm{Ord}^{\omega_1})$?) is universally Baire in $\mathcal{P}(\omega_1)$.

Preparation for generalization

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From now on, B will always be a complete Boolean algebra.

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Definition

The Stone space of B ($\mathrm{St}(B)$) is the collection of ultrafilters on B topologized by the sets of the form $O_b = \{G \in \mathrm{St}(B) \mid b \in G\}$ for some $b \in B$.

Remark

$\mathrm{St}(B)$ is compact Hausdorff.

Universally Baire subsets of 2^κ; Definitions

Convention

From now on,

- X will always be a topological space.
- κ will always be an infinite cardinal,
- we will identify $\mathcal{P}(\kappa)$ with $2^\kappa = \{x \mid x\colon \kappa \to 2\}$, and
- we will consider 2^κ as the product space of the discrete space $2 = \{0, 1\}$.

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Definition

Let $A \subseteq X$.

1. The set A is κ-meager if it is the union of κ-many nowhere dense sets in X.
2. The set A has the κ-Baire property in X if there is an open set U in X such that $U \triangle A = (U \setminus A) \cup (A \setminus U)$ is κ-meager.

Universally Baire subsets of 2^κ; Definitions ctd.

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Remark

1. If $X = \mathrm{St}(B)$ for some B, then X is κ-meager if and only if the forcing axiom for B with κ-many dense sets $\left(\mathrm{FA}_\kappa(B)\right)$ fails.

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Remark

1. If $X = \mathrm{St}(B)$ for some B, then X is κ-meager if and only if the forcing axiom for B with κ-many dense sets $(\mathrm{FA}_\kappa(B))$ fails.
2. If the whole space X is κ-meager, then every subset of X has the κ-Baire property in X.

Universally Baire subsets of 2^κ; Definitions ctd..

Definition

Let $A \subseteq 2^\kappa$. We say A is B-Baire in 2^κ if for any continuous $f : \mathrm{St}(B) \to 2^\kappa$, $f^{-1}(A)$ has the κ-Baire property in $\mathrm{St}(B)$.

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Remark

If $X = \mathrm{St}(B)$ is κ-meager (i.e., $\mathrm{FA}_\kappa(B)$ fails), then any subset of 2^κ is B-Baire.

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Remark

If $X = \mathrm{St}(B)$ is κ-meager (i.e., $\mathrm{FA}_\kappa(B)$ fails), then any subset of 2^κ is B-Baire.

Definition

Let Γ be a class of complete Boolean algebras and let A be a subset of 2^κ. We say A is universally Baire in 2^κ with respect to Γ ($\mathrm{uB}^\Gamma_\kappa$) if A is B-Baire in 2^κ for all B in Γ.

$\mathrm{uB}_\kappa^\Gamma$ sets; Examples

1. If $\kappa = \omega$ and Γ is the class of all complete Boolean algebras, then $\mathrm{uB}_\kappa^\Gamma$ sets are the same as universally Baire sets of reals due to Feng, Magidor, and Woodin.

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2. Let $\kappa = \omega_1$, $\mathbb{C}_{\omega_1}$ be the poset for adding ω_1-many Cohen reals, and Γ contain $\mathbb{C}_{\omega_1}$.
   Then if $\mathrm{FA}_{\omega_1}(\mathbb{C}_{\omega_1})$ holds, then no well-order on 2^{ω_1} is $\mathrm{uB}_\kappa^\Gamma$.

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   Then if $\mathrm{FA}_{\omega_1}(\mathbb{C}_{\omega_1})$ holds, then no well-order on 2^{ω_1} is $\mathrm{uB}^{\Gamma}_{\kappa}$.
3. (Caicedo and Velickovic)
   Let $\kappa = \omega_1$ and Γ be the class of complete Boolean algebras B such that B forces BPFA.
   Suppose that there are proper class many Woodin cardinals and assume that BPFA holds.
   Then there is a $\mathrm{uB}^{\Gamma}_{\omega_1}$ well-order on 2^{ω_1} which is Δ_1-definable in the structure $(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1})$ with a parameter of a subset of ω_1.

Universally Baire sets; Tree representation

Theorem (Feng, Magidor, and Woodin)

Let $A \subseteq 2^\omega$. Then the following are equivalent:

1. A is a universally Baire set of reals, and
2. for all B, there are a set Y and trees S_1, S_2 on $2 \times Y$ such that $A = \mathrm{p}[S_1]$ and $\Vdash_B$ "$\mathrm{p}[\check{S}_1] = 2^\omega \setminus \mathrm{p}[\check{S}_2]$".

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Notation

For b in B, let $B \upharpoonright b = \{c \in B \mid c \leq b\}$.

Theorem (Viale, I.)

Let B be such that $\mathrm{FA}_\kappa(B \upharpoonright b)$ holds for all b in B.
Then the following are equivalent:

1. A is B-Baire in 2^κ, and
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Notation

Let $[\kappa]^{<\omega}$ be the collection of finite subsets of κ.
For a set Y, let $\mathrm{Fn}(\kappa, Y) = \{s \mid s\colon \mathrm{dom}(s) \to Y \text{ and } \mathrm{dom}(s) \in [\kappa]^{<\omega}\}$.

Universally Baire sets; Tree representation ctd.

Recall (Theorem)

Let B be such that $\mathrm{FA}_\kappa(B \restriction b)$ holds for all b in B.
Then the following are equivalent:

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Definition

1. Let Y be a set. A subset S of $\mathrm{Fn}(\kappa, Y)$ is a tree$^\kappa$ on Y if S is closed under initial segments, i.e., if s is in S and $t \subseteq s$, then t is also in S.

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2. For a tree$^\kappa$ S on Y, $[S] = \{x \in Y^\kappa \mid (\forall u \in [\kappa]^{<\omega})\ x \upharpoonright u \in S\}$.

Universally Baire sets; Generic absoluteness

Theorem (Woodin)

Suppose there are proper class many Woodin cardinals.
Let $A \subseteq 2^\omega$. Then the following are equivalent:

1. A is universally Baire set of reals, and
2. there is a formula ϕ such that $(\forall \delta\colon \text{Woodin})$ $(\exists a\colon \text{set})$ such that
   - $A = \{x \in 2^\omega \mid \phi[x, a]\}$, and
   - for all B with $|B| < \delta$, all $\mathbb{Q}_{<\delta}$-generic H over V with $j\colon V \to M$, and all B-generic G over V with $G \in V[H]$, and for all $x \in 2^\omega \cap V[G]$,

$$V[G] \vDash \phi[x, a] \iff M \vDash \phi[x, j(a)].$$

Universally Baire sets; Generic absoluteness ctd.

Theorem (Viale, I.)

Suppose there are proper class many Woodin cardinals.
Let Γ be a class of B such that $\mathrm{FA}_\kappa(B \restriction b)$ holds for all b in B.
Then for all $A \subseteq 2^\kappa$, the following are equivalent:

1. A is $\mathrm{uB}^\Gamma_\kappa$, and
2. there is a formula ϕ such that $(\forall \delta > \kappa$: Woodin$)$ $(\exists a$: set$)$ such that
    - $A = \{x \in 2^\kappa \mid \phi[x, a]\}$, and
    - for all B in Γ with $|B| < \delta$, all $P^\kappa_{<\delta}$-generic H over V with $j \colon V \to M$, and all B-generic G over V with $G \in V[H]$, and for all $x \in 2^\kappa \cap V[G]$,

$$V[G] \vDash \phi[x, a] \iff M \vDash \phi[x, j(a)].$$

Universally Baire sets and iterable structures

Recall (Martin and Steel; Woodin)

Assuming the existence of proper class many Woodin cardinals and the generic nice UBH, for any countable $Z \prec_{\boldsymbol{\Sigma}_{2015}} V$, if M is the transitive collapse of Z, then there is an iteration strategy Σ for M such that Σ can be coded by a universally Baire set of reals.

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Theorem (Viale, I.)

Let $\kappa = \omega_1$ and Γ_0 be the class of all B such that $\mathsf{FA}_{\omega_1}(B \restriction b)$ holds for all b in B.

Assuming the existence of proper class many Woodin cardinals and the generic nice UBH, for any $Z \prec_{\mathbf{\Sigma}_{2015}} V$ of size ω_1 with $\omega_1 \subseteq Z$, if M is the transitive collapse of Z, then there is an iteration strategy Σ for M such that Σ can be coded by a $\mathrm{uB}^{\Gamma_0}_{\omega_1}$ set.

The End.