Automorphisms of $\mathcal{P}(\lambda)/I_\kappa$ for λ uncountable

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Given an infinite cardinal κ, we let I_κ denote the ideal of sets of cardinality less than κ.

$\mathrm{Fin} = I_{\aleph_0}$; $\mathrm{Ctble} = I_{\aleph_1}$

Given cardinals $\kappa \leq \lambda$ and $A \subseteq \lambda$, we let

$$[A]_{\lambda,\kappa} = \{B \subseteq \lambda \mid |A \triangle B| < \kappa\}$$

A function

$$\pi\colon \mathcal{P}(\lambda)/I_\kappa \to \mathcal{P}(\chi)/I_\rho$$

is said to be *trivial* on $A \subseteq \lambda$ if there exist $B \in [\lambda]^{<\kappa}$ and

$$f\colon A \setminus B \to \chi$$

such that

$$\pi([C]_{\lambda,\kappa}) = [f[C \setminus B]]_{\chi,\rho}$$

for all $C \subseteq A$.

The function π is *trivial* if it is trivial on λ.

Automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$

Theorem.(W. Rudin, 1956) Assuming CH there exist $2^{\aleph_1}$ many nontrivial automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$.

Theorem.(Shelah, late 1970's?) Consistently, all automorphisms of $\mathcal{P}(\omega)/\mathrm{Fin}$ are trivial.

Theorem.(Velickovic, 1993) Assuming PFA ($\mathrm{MA}_{\aleph_1}$ + OCA for $\lambda \leq \aleph_1$) all automorphisms of $\mathcal{P}(\lambda)/\mathrm{Fin}$ are trivial, for all cardinals λ.

Question 1. Is every automorphism of $\mathcal{P}(\lambda)/\mathrm{Fin}$ trivial on a cocountable set, for every uncountable cardinal λ?

Question 1a. Is every automorphism of $\mathcal{P}(\lambda)/\mathrm{Fin}$ trivial on an uncountable set, for every uncountable cardinal λ?

Question 2. Is every automorphism of $\mathcal{P}(\lambda)/I_\kappa$ trivial, whenever $\kappa \leq \lambda$ are uncountable?

Question 3. Must an automorphism of $\mathcal{P}(\lambda)/\mathrm{Fin}$ be trivial if it is trivial on all countable sets (or all sets of cardinality $\aleph_1$), for every uncountable cardinal λ?

A partial result on Questions 1 and 1a

Theorem.(Shelah-Steprāns) If $\lambda > 2^{\aleph_0}$ is less than the first strongly inaccessible cardinal, then every automorphism of $\mathcal{P}(\lambda)/\mathrm{Fin}$ is trivial on a subset of λ with complement of cardinality $2^{\aleph_0}$.

So : Question 1a has a positive answer for $\lambda > 2^{\aleph_0}$.

Question 4. (Turzanski/Katowice) Is is consistent that $\mathcal{P}(\omega)/\text{Fin}$ and $\mathcal{P}(\omega_1)/\text{Fin}$ are isomorphic?

Theorem.(Balcar-Frankiewicz) If λ and κ are distinct cardinals such that $\mathcal{P}(\kappa)/\text{Fin} \simeq \mathcal{P}(\lambda)/\text{Fin}$, then $\{\kappa, \lambda\} = \{\omega, \omega_1\}$.

One step of the proof shows that (**) if $\mathcal{P}(\omega)/\text{Fin}$ and $\mathcal{P}(\omega_1)/\text{Fin}$ are isomorphic then $\mathfrak{d} = \aleph_1$ (so $\text{MA}_{\aleph_1}$ fails).

The same proof shows that if

$$\kappa \leq \mu < \lambda$$

are cardinals (with κ regular) such that

$$\mathcal{P}(\mu)/I_\kappa \simeq \mathcal{P}(\lambda)/I_\kappa$$

then $\{\mu, \lambda\} = \{\kappa, \kappa^+\}$.

(Folklore) Suppose that $\mathcal{P}(\omega_1)/\mathrm{Fin}$ and $\mathcal{P}(\omega)/\mathrm{Fin}$ are isomorphic, and consider the automorphism (call it π) of $\mathcal{P}(\omega_1)/\mathrm{Fin}$ conjugate to the shift on ω. It has no nontrivial fixed points, which shows that it is not cocountably trivial.

Moreover, it has the property that for no (nontrivial) $A \subseteq^* B$ is $\pi([A]) = [B]$. (It has no nontrivial expanding points.)

Question 5. If π is an automorphism of $\mathcal{P}(\omega_1)/\mathrm{Fin}$, must there be an infinite, coinfinite $A \subset \omega_1$ such that $\pi([A]_{\mathrm{Fin}}) = [A]_{\mathrm{Fin}}$? That is, must π have a fixed point?

Question 6. If π is an automorphism of $\mathcal{P}(\omega_1)/\mathrm{Fin}$, must there be infinite, coinfinite $A \subseteq B \subset \omega_1$ such that $\pi([A]_{\mathrm{Fin}}) = [B]_{\mathrm{Fin}}$? That is, must π have an expanding point?

Theorem.(Hart) If $\mathcal{P}(\omega)/\mathrm{Fin}$ and $\mathcal{P}(\omega_1)/\mathrm{Fin}$ are isomorphic then there is a nontrivial automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$.

Proof: Break ω_1 into $\mathbb{Z}$-chains and consider the automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$ induced by shifting the chains. It has uncountably many minimal disjoint fixed points.

Question 7. Can there be an isomorphism from $\mathcal{P}(\omega_1)/\mathrm{Fin}$ to $\mathcal{P}(\omega)/\mathrm{Fin}$ which is trivial on all countable sets?

Preserving cardinalities

A function $\pi\colon \mathcal{P}(\lambda)/I_\kappa \to \mathcal{P}(\chi)/I_\rho$ is said to be *cardinality preserving* if for each $A \subseteq \lambda$ there exists a $B \subseteq \chi$ such that $|A| = |B|$ and $\pi([A]_{\lambda,\kappa}) = [B]_{\chi,\rho}$.

For any pairs of cardinals $\kappa < \lambda$ with κ regular, the existence of an isomorphism between $\mathcal{P}(\kappa^+)/I_\kappa$ and $\mathcal{P}(\kappa)/I_\kappa$ is equivalent to the existence of an automorphism of $\mathcal{P}(\lambda)/I_\kappa$ which is not cardinality preserving.

Selectors

A *selector* for a function

$$\pi\colon \mathcal{P}(\lambda)/I_\kappa \to \mathcal{P}(\chi)/I_\rho$$

is a function

$$\hat{\pi}\colon \mathcal{P}(\lambda) \to \mathcal{P}(\chi)$$

such that $\pi([A]_{\lambda,\kappa}) = [\hat{\pi}(A)]_{\chi,\rho}$ for all $A \subseteq \lambda$.

Lemma 1. Let $\kappa < \mu \leq \lambda$ be infinite cardinals, with κ regular, and let $\hat{\pi}$ be a selector for a cardinality preserving automorphism π of

$$\mathcal{P}(\lambda)/I_\kappa.$$

Define π^μ on $\mathcal{P}(\lambda)/I_\mu$ by setting

$$\pi^\mu([A]_{\lambda,\mu}) = [\hat{\pi}(A)]_{\lambda,\mu}.$$

Then π^μ is an automorphism of $\mathcal{P}(\lambda)/I_\mu$.

If π is not trivial on a set with compliment in I_μ, then π^μ is nontrivial.

Theorem 1. Let $\kappa \leq \mu$ be infinite cardinals, with κ is regular. Then every automorphism of $\mathcal{P}(2^\mu)/I_\kappa$ which is trivial on all sets of size μ^+ is trivial.

In the case $\kappa = \mu = \omega$: every automorphism of $\mathcal{P}(2^{\aleph_0})/\mathrm{Fin}$ which is trivial on all sets of cardinality at most $\aleph_1$ is trivial.

So $\mathrm{MA}_{\aleph_1}$ + OCA, which implies $2^{\aleph_0} = \aleph_2$ and (by Velickovic) that all automorphisms of $\mathcal{P}(\omega_1)/\mathrm{Fin}$ are trivial, implies (by Shelah-Steprāns) that all automorphisms of $\mathcal{P}(\lambda)/\mathrm{Fin}$ are trivial, for all λ below the least strongly inaccessible cardinal.

Proof of Theorem 1.

Let $\kappa \leq \mu$ be infinite cardinals, and suppose that π is an automorphism of $\mathcal{P}(2^\mu)/I_\kappa$.

Let $\hat{\pi}$ be a bijective selector for π, and let $\langle x_\beta : \beta < 2^\mu \rangle$ list $\mathcal{P}(\mu)$.

For each $\gamma < \mu$, let $R_\gamma = \{\beta < 2^\mu \mid \gamma \in x_\beta\}$.

For each $\alpha < 2^\mu$, let $y_\alpha = \{\gamma < \mu \mid \alpha \in \hat{\pi}^{-1}(R_\gamma)\}$.

Finally, for each $\alpha < 2^\mu$, let $h(\alpha) = \beta$ if $y_\alpha = x_\beta$.

For each $a \in [2^\mu]^{\leq\mu^+}$, let f_a be a trivializing function on a.

Then

$$|(h \restriction a) \triangle f_a| \leq \mu.$$

Assuming that π is not trivial, there exist pairwise disjoint a_α ($\alpha < \mu^+$) of cardinality at most κ^+ such that, for each α,

$$|(h \restriction a_\alpha) \triangle f_{a_\alpha}| \geq \kappa.$$

Let $a = \bigcup_{\alpha<\mu^+} a_\alpha$. Then

$$|(h \restriction a) \triangle f_a| \leq \mu$$

and, for all $\alpha < \mu^+$,

$$|(f_a \restriction a_\alpha) \triangle f_{a_\alpha}| < \kappa.$$

In fact, using the fact that every automorphism of $\mathcal{P}(\lambda)/I_\kappa$ is determined by how it acts on sets of size κ, one can show

Theorem 1′. For any infinite cardinal μ, every automorphism of $\mathcal{P}(2^\mu)/I_{\mu^+}$ which is trivial on all sets of size μ^+ is trivial.

What about automorphisms of $\mathcal{P}(2^{\aleph_0})/\mathrm{Fin}$ which are trivial on countable sets?

Given $\Gamma \subseteq \mathcal{P}(2^\omega)$, we let $\mathrm{CSN}(\Gamma)$ be the smallest cardinality of a family $F \subseteq (2^\omega)^\omega \times (2^\omega)^\omega$ such that

1. for every $(f,g) \in F$, $\{f(n) : n < \omega\} \cup \{g(n) : n < \omega\}$ is dense in 2^ω,
2. for all pairs $(f,g), (f',g')$ from F, if $g \neq g'$, then
$$\{g(n) : n < \omega\} \cap \{g'(n) : n < \omega\} = \emptyset,$$
3. for every $(f,g) \in F$ and $n < \omega$, $f(n) \neq g(n)$, and
4. for every set $A \in \Gamma$, the set
$$\{(f,g) \in F : \exists^\infty n < \omega \ |A \cap \{f(n), g(n)\}| = 1\}$$
has cardinality smaller than that of F,

if such a family F exists. If no such family exists, we set $\mathrm{CSN}(\Gamma) = (2^{\aleph_0})^+$.

CSN(open) $\geq$ cov(Meager)

Theorem 2. If CSN(Borel) $> \aleph_1$, then every cardinality preserving automorphism of $\mathcal{P}(2^{\aleph_0})/\mathrm{Fin}$ which is trivial on all countable sets is trivial.

By Theorem 1, it suffices to show this for automorphisms of $\mathcal{P}(\omega_1)/\mathrm{Fin}$.

Suppose that π is a cardinality-preserving automorphism of $\mathcal{P}(\omega_1)/\mathrm{Fin}$ which is trivial on countable sets.

Let $\hat{\pi}$ be a bijective selector for π.

Let x_α $(\alpha < \omega_1)$ be distinct subsets of ω.

For each $n < \omega$, let $R_n = \{\alpha < \omega_1 \mid n \in x_\alpha\}$.

For each $\alpha < \omega_1$, let $y_\alpha = \{n < \omega \mid \alpha \in \hat{\pi}^{-1}(R_n)\}$.

Finally, for each $\alpha < \omega_1$, let $h(\alpha) = \beta$ if $y_\alpha = x_\beta$ (and 0 if there is no such β).

Let f_α $(\alpha < \omega_1)$ be functions witnessing that π is trivial on each α.

Assuming that π is not trivial, we get an increasing sequence $\langle \alpha_\xi : \xi < \omega_1 \rangle$ of elements of ω_1 such that for each ξ there exists

$$\{\beta_i^\xi : i < \omega\} \subseteq [\alpha_\xi, \alpha_{\xi+1})$$

such that the sequences $\langle h(\beta_i^\xi) : i < \omega \rangle$ and $\langle f_{\alpha_{\xi+1}}(\beta_i^\xi) : i < \omega \rangle$ satisfy conditions 1-3 in the defintion of CSN.

For every Borel set $B \subseteq \mathcal{P}(\omega)$, letting $A_B = \{\alpha \mid x_\alpha \in B\}$, $h^{-1}[A_B] \triangle \hat{\pi}^{-1}(A_B)$ is countable.

Applying CSN(Borel) then gives a contradiction.

A partial result on Question 2

A Q_B-set is a set $A \subset \mathcal{P}(\omega)$ such that for all $X \subseteq A$ there is a Borel $B \subseteq \mathcal{P}(\omega)$ such that $X = A \cap B$. The set A is a Q-set if the set B can be taken to be F_σ.

Martin Axiom for partial orders of cardinality κ (MA_κ) implies that every subset of $\mathcal{P}(\omega)$ of cardinality κ is a Q-set (i.e., $\mathfrak{q}_0 > \kappa$). We let $\mathfrak{z}$ be the least cardinality of a non-Q_B-set.

Theorem 3. Given the existence of one Q_B-set (of cardinality λ), the existence of a nontrivial automorphism of $\mathcal{P}(\lambda)/\text{Ctble}$ is equivalent to the existence of two disjoint Q_B-sets of cardinality at most λ which intersect the same Borel sets uncountably.

So, if $\mathfrak{z} > \lambda$, then every automorphism of $\mathcal{P}(\lambda)/\text{Ctble}$ is trivial.

Proof of Theorem 3.

Let $X = \{x_\beta : \beta < \lambda\}$ be a Q_B-set, and suppose that π is an automorphism of $\mathcal{P}(\lambda)/\mathrm{Ctble}$. Let $\hat{\pi}$ be a bijective selector for π.

For each $n \in \omega$, let $R_n = \{\beta < \lambda \mid n \in x_\beta\}$.

For each $\alpha < \lambda$, let $y_\alpha = \{n < \omega \mid \alpha \in \hat{\pi}^{-1}(R_n)\}$.

Then $Y = \{y_\alpha : \alpha < \lambda\}$ is a Q_B-set, and for each Borel set B,

$$\pi([\{\alpha < \lambda : y_\alpha \in Y \cap B\}]) = [\{\alpha < \lambda : x_\alpha \in X \cap B\}].$$

Letting $Z = X \cap Y$, and defining h on Z by setting $h(\alpha) = \beta$ if $y_\alpha = x_\beta$, we have that h witnesses that π is trivial on Z, and that π is nontrivial on $Y \setminus Z$. Furthermore, $Y \setminus Z$ and $X \setminus Z$ intersect the same Borel sets uncountably.

Putting together Theorem 3 with Lemma 1, (**) and the fact that $\mathfrak{q}_0 \leq \mathfrak{d}$ we get that if $\mathfrak{q}_0 > \aleph_1$, then every automorphism of $\mathcal{P}(\mathbb{R})/\text{Fin}$ is trivial on a cocountable set (again, this extends to all λ less than the least strongly inaccesible cardinal).

We don't know if we can replace $\mathfrak{q}_0$ with $\mathfrak{z}$ here (i.e., whether $\mathfrak{z} > \aleph_1$ implies that $\mathcal{P}(\omega_1)/\text{Fin}$ and $\mathcal{P}(\omega)/\text{Fin}$ are non-isomorphic).

Velickovic has shown that $\text{MA}_{\aleph_1}$ is consistent with the existence of nontrivial automorphisms of $\mathcal{P}(\omega)/\text{Fin}$.

Fixed points

Lemma. Suppose that $\kappa \leq \lambda$ are infinite cardinals, and that π is an automorphism of $\mathcal{P}(\lambda)/I_\kappa$. Let π^* be a selector for π. Let η be an infinite regular cardinal unequal to $cf\,\kappa$, and suppose that $\langle A_\alpha : \alpha < \eta \rangle$ is a sequence of subsets of λ such that

$$|(A_\alpha \cup \pi^*(A_\alpha) \cup (\pi^*)^{-1}(A_\alpha)) \setminus A_\beta| < \kappa$$

for all $\alpha < \beta < \eta$. Then $\bigcup\{A_\alpha : \alpha < \eta\}$ is a fixed point of π.

It follows that π has nontrivial fixed points if either

- κ is regular and $\lambda > \kappa^+$, or
- κ is uncountable and π is cardinality preserving.

If κ is regular, π is a cardinality-preserving automorphism of

$$\mathcal{P}(\kappa^+)/I_\kappa$$

and S is the set of non-fixed points of π, then

- nonstationarily many members of S have cofinality less than κ;
- there exist a club $C \subseteq \kappa^+$ and a covering of $C \cap S$ by two subsets, each of which carry a ladder system for which 2-uniformization holds.

Theorem.(Devlin-Shelah) If there is a partition of club subset of ω_1 into two sets for which 2-uniformization holds, then $2^{\aleph_0} = 2^{\aleph_1}$.

So : the existence of a cardinality-preserving automorphism of $\mathcal{P}(\omega_1)/\mathrm{Fin}$ without nontrivial fixed points implies $2^{\aleph_0} = 2^{\aleph_1}$.

Uniformization

Say that a collection of sets X is strongly uniformized if whenever $y_x \subseteq x$ for each $x \in X$, there is a set z such that $z \cap x =^* y_x$ for each $x \in X$.

For a ladder system, being strongly uniformized is the same as 2-uniformization.

A strong Q-set is a strongly uniformized MAD family in $\mathcal{P}(\omega)$ (Steprans has shown that these are consistent with MA(σ-centered) but not MA(σ-linked)).

Given a set A, say that a collection of sets $X = \{x_a : a \in A\}$ is an A-strongly-uniformized if

- whenever $a, b \in A$ are such that $a \subseteq b$ and $b \setminus a$ is infinite, $x_a \subseteq^* x_b$ and $x_b \setminus x_a$ is infinite;
- whenever $B \subseteq A$ is pairwise disjoint, $\{x_a : a \in B\}$ is strongly uniformized.

If $\hat{\pi}$ is a selector for an isomorphism from $\mathcal{P}(\omega_1)/\text{Fin}$ to $\mathcal{P}(\omega)/\text{Fin}$, then $\{\hat{\pi}(a) : a \subseteq \omega_1\}$ is a $\mathcal{P}(\omega_1)$-strongly uniformized set.

If T is either of the two sets in the covering of $C \cap S$ as above, then there exists a $\mathcal{P}(T)$-strongly uniformized set $\{x_a : a \subseteq T\}$ such that for each $\alpha \in T$, $x_{\{\alpha\}}$ is a cofinal subset of α.

Question 8. Can either of these things exist?