The bi-embeddability relation on countable torsion-free abelian groups

Filippo Calderoni

University of Turin

Descriptive Set Theory in Turin
September 8, 2017

Notation

- "countable" $=$ "countably infinite";

Notation

- "countable" = "countably infinite";
- a **graph** is an irreflexive and symmetric binary relation;

Notation

- "countable" = "countably infinite";
- a **graph** is an irreflexive and symmetric binary relation;
- Let X_{Gr} be the **Polish space of countable graphs** (with vertex-set ω).

Notation

- "countable" = "countably infinite";
- a **graph** is an irreflexive and symmetric binary relation;
- Let X_{Gr} be the **Polish space of countable graphs** (with vertex-set ω).

Definition

Let $T, U \in X_{Gr}$.

$$T \sqsubseteq_{Gr} U \quad \overset{def}{\iff} \quad \exists h \in \omega^{\omega} \,(f \text{ is an isomorphism from } T \text{ to } U \upharpoonright Im(f)).$$

$$T \equiv_{Gr} U \quad \overset{def}{\iff} \quad T \sqsubseteq_{Gr} U \text{ and } U \sqsubseteq_{Gr} T.$$

Notation

- "countable" = "countably infinite";
- a **graph** is an irreflexive and symmetric binary relation;
- Let X_{Gr} be the **Polish space of countable graphs** (with vertex-set ω).

Definition

Let $T, U \in X_{Gr}$.

$$T \sqsubseteq_{Gr} U \quad \stackrel{def}{\Longleftrightarrow} \quad \exists h \in \omega^\omega \,(f \text{ is an isomorphism from } T \text{ to } U \restriction Im(f)).$$

$$T \equiv_{Gr} U \quad \stackrel{def}{\Longleftrightarrow} \quad T \sqsubseteq_{Gr} U \text{ and } U \sqsubseteq_{Gr} T.$$

$\sqsubseteq_{Gr}$ and $\cong_{Gr}$ are $\mathbf{\Sigma}^1_1$ subsets of $X_{Gr} \times X_{Gr}$.

Notation

In a similar way we define $\sqsubseteq_{Gp}, \equiv_{Gp}$ on the **Polish space of countable groups** X_{Gp}.

Notation

In a similar way we define $\sqsubseteq_{Gp}, \equiv_{Gp}$ on the **Polish space of countable groups** X_{Gp}.

Definition

Let Q, R be quasi-orders on the standard Borel spaces X, Y, respectively. We say that Q **Borel reduces** to R ($Q \leq_B R$) if there exists a Borel $f : X \to Y$ such that for all $x, y \in X$

$$x \, Q \, y \iff f(x) \, R \, f(y).$$

Starting point

Theorem (J. Williams 2014)

The relation $\equiv_{Gp}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation. That is, whenever E is a $\mathbf{\Sigma}^1_1$ equivalence relation $E \leq_B \equiv_{Gp}$.

Starting point

Theorem (J. Williams 2014)

The relation $\equiv_{Gp}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation. That is, whenever E is a $\mathbf{\Sigma}^1_1$ equivalence relation $E \leq_B \equiv_{Gp}$.

Proof (outline)

By producing a Borel map $f\colon X_{Gr} \to X_{Gp}$ such that

$$T \sqsubseteq_{Gr} U \iff f(T) \sqsubseteq_{Gp} f(U).$$

Starting point

Theorem (J. Williams 2014)

The relation $\equiv_{Gp}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation. That is, whenever E is a $\mathbf{\Sigma}^1_1$ equivalence relation $E \leq_B \equiv_{Gp}$.

Proof (outline)

By producing a Borel map $f\colon X_{Gr} \to X_{Gp}$ such that

$$T \sqsubseteq_{Gr} U \quad \iff \quad f(T) \sqsubseteq_{Gp} f(U).$$

It follows that $\equiv_{Gp}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation. □

For every $T \in X_{Gr}$, the group $f(T)$ is **non-abelian** and have **many torsion elements**, which are used to encode the edge-relation.

For every $T \in X_{Gr}$, the group $f(T)$ is **non-abelian** and have **many torsion elements**, which are used to encode the edge-relation.

Question

What is the Borel complexity of the bi-embeddability relation $\equiv_{TFA}$ on the space of countable **torsion-free abelian** group?

Recall that an abelian group $(G, +, 0)$ is **torsion-free** if

$$\forall g \in G \ \forall n \in \mathbb{N} \setminus \{0\}(ng = 0 \to g = 0).$$

A first possible strategy

Theorem (Przeździecki 2014)

There exists an **almost-full embedding** G *from* $\mathcal{G}raphs$ *into* $\mathcal{A}b$. *That is, for every two graphs* T, V

$$\mathbb{Z}[\mathrm{Hom}(T, V)] \cong \mathrm{Hom}(GT, GV).$$

$\mathbb{Z}[S]$ is the free abelian group generated by the set S.

A first possible strategy

Theorem (Przeździecki 2014)

There exists an **almost-full embedding** G *from* $\mathcal{G}raphs$ *into* $\mathcal{A}b$. *That is, for every two graphs* T, V

$$\mathbb{Z}[\mathrm{Hom}(T, V)] \cong \mathrm{Hom}(GT, GV).$$

$\mathbb{Z}[S]$ is the free abelian group generated by the set S.

Every group in the target is actually torsion-free, and G preserves injectiveness.

A "generalized" result

By slightly modifying Przeździecki's functor we have

Theorem (C.)

If κ is uncountable and $\kappa^{<\kappa} = \kappa$, then $\sqsubseteq^{\kappa}_{Gr}$ Borel reduces to $\sqsubseteq^{\kappa}_{TFA}$.

A "generalized" result

By slightly modifying Przeździecki's functor we have

Theorem (C.)

If κ is uncountable and $\kappa^{<\kappa} = \kappa$, then $\sqsubseteq^{\kappa}_{Gr}$ Borel reduces to $\sqsubseteq^{\kappa}_{TFA}$.

Combined with (Motto Ros 2013; Mildenberger-Motto Ros)...

Corollary

If κ is uncountable and $\kappa^{<\kappa} = \kappa$, then the relation $\equiv^{\kappa}_{TFA}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

A “generalized” result

By slightly modifying Przeździecki’s functor we have

Theorem (C.)

If κ is uncountable and $\kappa^{<\kappa} = \kappa$, then $\sqsubseteq^{\kappa}_{Gr}$ Borel reduces to $\sqsubseteq^{\kappa}_{TFA}$.

Combined with (Motto Ros 2013; Mildenberger-Motto Ros)...

Corollary

If κ is uncountable and $\kappa^{<\kappa} = \kappa$, then the relation $\equiv^{\kappa}_{TFA}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

The functor cannot be used in the classical case because it maps countable graphs to groups of size $2^{\aleph_0}$.

Completeness of $\equiv_{TFA}$

Theorem (C.-Thomas)

The bi-embeddabilty relation $\equiv_{TFA}$ on the space of countable torsion free abelian group is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

Completeness of $\equiv_{TFA}$

Theorem (C.-Thomas; Törnquist)

The bi-embeddabilty relation $\equiv_{TFA}$ on the space of countable torsion free abelian group is a complete $\boldsymbol{\Sigma}^1_1$ equivalence relation.

Completeness of $\equiv_{TFA}$

Theorem (C.-Thomas; Törnquist)

The bi-embeddabilty relation $\equiv_{TFA}$ on the space of countable torsion free abelian group is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

Proof (outline)

Louveau-Rosendal 2005 defined a complete $\mathbf{\Sigma}^1_1$ quasi-order $\leq_{max}$ on a Polish space $\mathcal{T}$ of trees on $2 \times \omega$ (normal trees).

Completeness of $\equiv_{TFA}$

Theorem (C.-Thomas; Törnquist)

The bi-embeddabilty relation $\equiv_{TFA}$ on the space of countable torsion free abelian group is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

Proof (outline)

Louveau-Rosendal 2005 defined a complete $\mathbf{\Sigma}^1_1$ quasi-order $\leq_{max}$ on a Polish space $\mathcal{T}$ of trees on $2 \times \omega$ (normal trees).

We define a reduction of $\leq_{max}$ to $\sqsubseteq_{TFA}$ by composition

$$T \longmapsto G_T \longmapsto A(G_T).$$

Where G_T is the combinatorial tree built from T as in (Louveau-Rosendal 2005). And $A(G_T)$ is an adaptation of the torsion-free abelian group built from G_T as in (Downey-Montalban 2008).

We prove that

$$T \leq_{max} U \quad \iff \quad A(G_T) \sqsubseteq_{TFA} A(G_U).$$

It follows that $\equiv_{TFA}$ is a complete $\boldsymbol{\Sigma}^1_1$ equivalence relation. □

Classical vs. generalized DST

Theorem (C.-Thomas; Törnquist)

The bi-embeddabilty relation $\equiv_{TFA}$ on the space of countable torsion-free abelian group is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

Theorem (C.)

If κ is uncountable and $\kappa^{<\kappa} = \kappa$, then $\equiv^{\kappa}_{TFA}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

The proofs use essentially different techniques.

Torsion-free vs. torsion

Theorem (C.-Thomas; Törnquist)

The bi-embeddabilty relation $\equiv_{TFA}$ on the space of countable torsion free abelian group is a complete $\boldsymbol{\Sigma}^1_1$ equivalence relation.

Consider X_{TA} the space of torsion abelian group.

Theorem (C.-Thomas)

The equivalence relations $\equiv_{\mathrm{TA}}$ and $\cong_{\mathrm{TA}}$ are incomparable up to Borel reducibility.

Invariant universality

Theorem (C.-Motto Ros)

For every $\boldsymbol{\Sigma}^1_1$ equivalence relation E, there exists a Borel $B_E \subseteq X_{Gp}$ such that

- B_E *is* $\cong_{Gp}$*-invariant*
- $E \sim (\equiv_{Gp} \restriction B_E)$.

Invariant universality

Theorem (C.-Motto Ros)

For every $\mathbf{\Sigma}^1_1$ equivalence relation E, there exists a Borel $B_E \subseteq X_{Gp}$ such that

- B_E *is* $\cong_{Gp}$*-invariant*
- $E \sim (\equiv_{Gp} \restriction B_E)$.

Using the terminology of (Camerlo-Marcone-Motto Ros 2013), $(\equiv_{Gp}, \cong_{Gp})$ is **invariantly universal**.

Invariant universality

Theorem (C.-Motto Ros)

For every $\mathbf{\Sigma}^1_1$ equivalence relation E, there exists a Borel $B_E \subseteq X_{Gp}$ such that

- B_E *is* $\cong_{Gp}$*-invariant*
- $E \sim (\equiv_{Gp} \restriction B_E)$.

Using the terminology of (Camerlo-Marcone-Motto Ros 2013), $(\equiv_{Gp}, \cong_{Gp})$ is **invariantly universal**.

Corollary

For every $\mathbf{\Sigma}^1_1$ equivalence relation E, there exists a sentence $\varphi \in \mathcal{L}_{\omega_1\omega}$ such that $E \sim \equiv_{Gp} \restriction \mathrm{Mod}_\varphi$.

How about $\equiv_{TFA}$?

Conjecture

$(\equiv_{TFA}, \cong_{TFA})$ is invariantly universal.

Impossible not to mention...

Theorem (Hjorth 2002)

The isomorphism relation $\cong_{TFA}$ is not Borel.

Impossible not to mention...

Theorem (Hjorth 2002)

The isomorphism relation $\cong_{TFA}$ is not Borel.

Theorem (Downey-Montalban 2008)

The isomorphism relation $\cong_{TFA}$ is a complete $\mathbf{\Sigma}_1^1$ set.

Impossible not to mention...

Theorem (Hjorth 2002)

The isomorphism relation $\cong_{TFA}$ is not Borel.

Theorem (Downey-Montalban 2008)

The isomorphism relation $\cong_{TFA}$ is a complete $\mathbf{\Sigma}^1_1$ set.

Question

Is $\cong_{TFA}$ a complete S_∞-equivalence relation?

A good conjecture

Suspect (Törnquist)

$\cong_{TFA}$ is NOT a complete S_∞-equivalence relation.

A good conjecture

Conjecture (Törnquist)

$\cong_{TFA}$ is NOT a complete S_∞-equivalence relation.

What is a good conjecture?

> *" The most interesting statement as possible...which is not provably false."*
>
> *-Simon Thomas*