Generic unitary representations

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Introduction-the full generality

Let Γ be a countable discrete group and G a topological group. The set of all homomorphisms of Γ into G may be identified with a closed subspace, denoted by $\mathrm{Rep}(\Gamma, G)$, of the topological space G^Γ.

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This has been investigated for example when $G = \mathrm{GL}(n, \mathbb{K})$ or $G = U(H)$. More generally, recently it has been considered for $G = \mathrm{Aut}(X)$, where X is some countable structure, e.g. set, graph, etc.

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If G is Polish, then $\mathrm{Rep}(\Gamma, G)$ is also Polish, thus a Baire space and one may consider properties of $\mathrm{Rep}(\Gamma, G)$ that are satisfied by meager, resp. comeager many elements.

Generic homomorphisms

Of particular interest is the question whether there are generic homomomorphisms. Call two homomorphisms $\pi_1, \pi_2 \in \mathrm{Rep}(\Gamma, G)$ *equivalent* if there is $g \in G$ such that

$$\pi_1(x) = g \cdot \pi_2(x) \cdot g^{-1}$$

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Theorem (Y. Glasner, Kitroser, Melleray, 2016)

A countable discrete Γ has a generic permutation representation (i.e. comeager class in $\mathrm{Rep}(\Gamma, S_\infty)$) iff Γ is solitary (LERF implies solitary).

Theorem (Rosendal, 2011)

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Theorem (Del Junco? Rokhlin?)

Every conjugacy class in $U(H)$ is meager.

Generic homomorphisms

Notice that $U(H)$ is naturally homeomorphic with $\mathrm{Rep}(\mathbb{Z}, H)$ (analogously, $U(H)^n$ is naturally homeomorphic with $\mathrm{Rep}(F_n, H)$). So it follows and is known:

Theorem

If Γ is a finitely generated free group, then all equivalence classes in $\mathrm{Rep}(\Gamma, H)$ are meager.

Main results

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Let Γ be a countable discrete group and suppose that finite-dimensional unitary representations are dense in $\hat{\Gamma}$ (equivalently, finite-dimensional representations are dense in $\mathrm{Rep}(\Gamma, H)$).
Then we have

- if Γ is infinite and has the Haagerup property, then the equivalence classes in $\mathrm{Rep}(\Gamma, H)$ are meager;
- if Γ has Kazhdan's property T, then there is a comeager equivalence class in $\mathrm{Rep}(\Gamma, H)$

Positive definite functions

Let $\pi \in \mathrm{Rep}(\Gamma, H)$ and $\xi \in H$ is a unit vector. A function $\phi : \Gamma \to \mathbb{C}$ is *normalized positive definite* if it is of the form $g \to \langle \pi(g)\xi, \xi \rangle$.

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Definition

Γ has the Haagerup property (or is a-T-menable) if there exists a sequence $(\phi_n)_n$ of normalized positive definite functions on Γ such that

- they vanish at infinity, i.e. $(\phi_n)_n \subseteq c_0(\Gamma)$;
- they converge pointwise to the constant function 1.

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- they vanish at infinity, i.e. $(\phi_n)_n \subseteq c_0(\Gamma)$;
- they converge pointwise to the constant function 1.

Equivalently, Γ admits a proper action on a Hilbert space by isometries.

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Theorem

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Idea of the proof.

For a fixed countable dense subset D of the unit sphere in H and for any normalized positive definite function ϕ on Γ, the set $I_\phi = \{\pi \in \mathrm{Rep}(\Gamma, H) : \forall \xi \in D\ \exists x \in \Gamma\ (|\phi(x) - \phi_{\pi,\xi}(x)| > 1/4\}$ is dense G_δ.

Kazhdan's property

Definition

A countable discrete group Γ has the Kazhdan's property T if there are a finite set $F \subseteq \Gamma$ and $\varepsilon > 0$ such that whenever $\pi \in \mathrm{Rep}(\Gamma, H)$ has an (F, ε)-almost invariant unit vector, then it has an invariant vector.
Equivalently, if $1_\Gamma \preceq \pi$, then $1_\Gamma \leq \pi$.
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Fact

Invariant vectors are 'close' to the almost invariant ones. That is, if $\xi \in H$ is a unit $(F, \delta \cdot \varepsilon)$-almost invariant vector for $\pi \in \mathrm{Rep}(\Gamma, H)$, then there is an invariant vector $\xi' \in H$ such that $\|\xi - \xi'\| < \delta$.

Theorem (Wang)

If Γ has the Kazhdan's property and σ is a finite-dimensional irreducible unitary representation of Γ, then for any $\pi \in \mathrm{Rep}(\Gamma, H)$ we have that if $\sigma \preceq \pi$, then $\sigma \leq \pi$.

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Theorem

Let σ be a finite-dimensional irreducible representation of a Kazhdan group Γ. Let $\pi \in \mathrm{Rep}(\Gamma, H)$ and $\xi \in H$, $\|\xi\| = 1$, be such that 'locally, σ and π behave on ξ almost the same'. Then there exists $\xi' \in H$ 'close' to ξ such that the subrepresentation of π induced by ξ' is equivalent to σ.

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Theorem

Let Γ be a countable discrete Kazhdan group with finite-dimensional representations dense. Then Γ has a generic representation, i.e. a representation with a comeager equivalence class. It is the direct sum of all finite-dimensional irreducible representations, each with infinite multiplicity.

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Let Γ be a countable discrete Kazhdan group with finite-dimensional representations dense. Then Γ has a generic representation, i.e. a representation with a comeager equivalence class. It is the direct sum of all finite-dimensional irreducible representations, each with infinite multiplicity.

The previous theorem allows us to compute that the conjugacy class is G_δ and the assumption that finite-dimensional representations dense in $\hat{\Gamma}$ allows us to show that this conjugacy class is dense.

Open question - Bekka, de la Harpe, Valette

Does there exist an infinite group Γ with the Kazhdan property such that finite-dimensional representations are dense in $\hat{\Gamma}$?

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Does there exist a Kazhdan group with property FD?

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Property FD of Γ is stronger than that finite-dimensional representations are dense in $\hat{\Gamma}$.

Question

Does there exist an infinite Kazhdan group Γ such that isolated points in $\hat{\Gamma}$ are dense?

Representations of C*-algebras

Let A be a separable (unital) C*-algebra. Then the set of all representations of A in $B(H)$ is a Polish space (a Polish subset of the non-metrizable space $B(H)^A$).

Facts

- For a countable discrete group Γ the spaces $\mathrm{Rep}(\Gamma, H)$ and $\mathrm{Rep}(C^*(\Gamma), H)$ are naturally homeomorphic.
- (Archbold; Exel and Loring) A C*-algebra A is residually finite-dimensional iff finite-dimensional representations are dense in $\mathrm{Rep}(A, H)$ iff finite-dimensional representations are dense in $\hat{A}$.

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Restatement of the theorem

Let Γ be a countably infinite discrete group with the Haagerup property and suppose that the full group C*-algebra $C^*(\Gamma)$ is residually finite-dimensional. Then all equivalence classes in $\mathrm{Rep}(C^*(\Gamma), H)$ are meager.

Representations of C*-algebras

Theorem

Let A be a separable unital and infinite-dimensional abelian C*-algebra such that $\hat{A}$ doesn't have isolated points. Then all equivalence classes in $\mathrm{Rep}(A, H)$ are meager.

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Let A be a separable unital and infinite-dimensional abelian C*-algebra such that $\hat{A}$ doesn't have isolated points. Then all equivalence classes in $\mathrm{Rep}(A, H)$ are meager.

Idea of the proof.

Fix a countable dense subset D of the unit sphere in H and a countable dense subset Γ of the unit sphere in A. Then for any state ϕ on A, the set
$I_\phi = \{\pi \in \mathrm{Rep}(A, H) : \forall \xi \in D\, \exists x \in \Gamma\, (|\phi(x) - \phi_{\pi,\xi}(x)| > 1/4\}$ is dense G_δ.

Representations of C*-algebras

Theorem

If (A, α, Γ) is a C*-dynamical system, where Γ is a countable discrete group with the Haagerup property and $B = C^*(A, \alpha, \Gamma)$ is residually finite-dimensional. Then all equivalence classes in $\mathrm{Rep}(B, H)$ are meager.

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If (A, α, Γ) is a C*-dynamical system, where Γ is a countable discrete group with the Haagerup property and $B = C^*(A, \alpha, \Gamma)$ is residually finite-dimensional. Then all equivalence classes in $\mathrm{Rep}(B, H)$ are meager.

Corollary

Suppose that $\Gamma = \Delta \rtimes_\alpha \Lambda$, where Λ is an infinite group with the Haagerup property and α an action of Λ on Δ, and $C^*(\Gamma)$ is residually finite-dimensional. Then the conjugacy classes in $\mathrm{Rep}(C^*(\Gamma), H)$ (or $\mathrm{Rep}(\Gamma, H)$) are meager.