Descriptive and combinatorial set theory at singular cardinals and their successors

Mirna Džamonja

School of Mathematics, University of East Anglia,
associée IHPST, Université Panthéon-Sorbonne, Paris 1

Torino, September 2017

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These authors have developed a rich theory, mostly concentrating on the case κ regular, in particular successor of regular or inaccessible. Often, the generalised Baire space does not allow direct generalisations of theorems about the Baire space and new techniques and expectations have to be made.

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A set $A \subseteq \kappa^\omega$ is Π^1_1 if there is an open set $B \subseteq \kappa^\omega \times \kappa^\omega$ (in the product topology) such that for every $f \in {}^\omega\kappa$

$$f \in A \iff \forall g\, ((f, g)) \in B). \tag{1}$$

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A set is Σ^1_1 if its complement is Π^1_1 and it is Δ^1_1 if it is both Π^1_1 and Σ^1_1.

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Theorem (Boundedness Theorem)

Suppose that A is Π^1_1 in κ^ω and represented by the tree T. Then A is Δ^1_1 if and only if there is $g \in \mathcal{TO}$ such that $\forall f \in A (T(f) \leq T(g))$.

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For $f \in {}^{\kappa}\kappa$ let $g_f \in {}^{\omega}\omega$ be given by $g_f(n) = \mathbf{k}(f(\kappa_n))$. For $g \in {}^{\omega}\omega$ let $f^g \in {}^{\kappa}\kappa$ be given by letting for all α,

$$f^g(\alpha) = \kappa_{n+1} \text{ iff } g(\mathbf{k}(\alpha)) = n.$$

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We note that generalised cardinal invariants for regular cardinals can behave quite wildly, this is well documented in works by various authors.

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Theorem

Modulo large cardinals, it is consistent to have a singular cardinal κ with countable cofinality such that the density of the countably supported box product space is $\kappa^+ < 2^\kappa$.

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2. get consistency results about cardinal invariants at a large cardinal, obtained by Garti and Shelah and Brooke-Taylor, V. Fischer, S. Friedman and Montoya.

For example, what can be said about the generalised Baire space at $\lambda = \kappa^+$?

Recent progress

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