# The Consistency of Uniformisation

Important consequences of PD are:

Projective Regularity: Projective sets are measurable and have the Baire property

Projective Uniformisation: Projective binary relations can be uniformised by projective functions

Woodin conjectured that these properties together are as strong as PD, but Steel showed that they can be obtained from less. Woodin's conjecture was in principle correct as it does hold with "projective" replaced by "hyperprojective" and other sufficiently rich pointclasses.

# The Consistency of Uniformisation

Projective regularity and Projective Uniformisation are individually rather weak:

Projective regularity needs only an inaccessible (Solovay's model)

Projective Uniformisation holds in $L$.

But in fact PD gives something stronger than Projective Uniformisation:

Moschovakis Uniformisation: $\Pi^1_n$ binary relations can be uniformised by $\Pi^1_n$ functions for odd $n$.

This does *not* hold in $L$.

*Question.* What is the strength of $\Pi^1_3$ Uniformisation?

# The Consistency of Uniformisation

Harrington obtained numerous consistency results regarding projective sets. In particular he showed that the following is consistent relative to ZFC:

Moschovakis Reduction: $\Pi^1_n$ has the reduction property for each odd $n$.

($\Pi^1_n$ reduction says that any two $\Pi^1_n$ sets have $\Pi^1_n$ subsets which are disjoint with the same union.)

But he left the consistency of Moschovakis Uniformisation as an open problem.

# The Consistency of Uniformisation

**Theorem**

*(SDF-Hoffelner) Relative to ZFC it is consistent that $\Pi^1_3$ uniformsation holds.*

It looks like the proof would extend to yield the consistency of $\Pi^1_5$ uniformsation relative to ZFC + 2 Woodin cardinals. But in fact I conjecture:

*Conjecture.* $\Pi^1_n$ Uniformisation for odd $n$ is consistent relative to just ZFC.

# The Consistency of Uniformisation

The above result about $\Pi^1_3$ uniformisation was in fact proved in response to a question of Vassilis Gregoriades about generalised Baire space.

Let $BS(\omega_1)$ denote the generalised Baire space $\omega_1^{\omega_1}$ topologised with basic open sets $\{f \mid \eta \subseteq f\}$ where $\eta$ is from $\omega_1^{<\omega_1}$. Borel sets are obtained from these by closing under unions and intersections of size $\omega_1$ as well as complements. Projective sets are then formed, well, by projecting. Meager sets are $\omega_1$-unions of nowhere dense sets. The Baire property means the usual thing.

# The Consistency of Uniformisation

One can consider:

Projective Regularity for $BS(\omega_1)$: Projective sets have the Baire property.

Moschovakis Uniformisation for $BS(\omega_1)$: $\Pi^1_n$ binary relations can be uniformised by $\Pi^1_n$ functions for odd $n$.

Projective Regularity for $BS(\omega_1)$ provably *fails*: The Club filter is projective and does not have the Baire property.

# The Consistency of Uniformisation

Vassilis asked about the consistency of Moschovakis Uniformisation.

**Theorem**

*(SDF-Hoffelner) Relative to ZFC it is consistent that CH holds and $\Pi^1_1$ uniformsation holds for $BS(\omega_1)$.*

The above-mentioned result about $\Pi^1_3$ uniformisation is an adaptation of the proof of the above result.

## The Consistency of Uniformisation

On the proof of the consistency of $\Pi^1_1$ Uniformisation for $BS(\omega_1)$:

We start with $L$ as ground model and perform a proper, $\omega_2$-cc iteration of length $\omega_2$ that does not add reals.

Let $\varphi(x, y)$ be a universal $\Pi_1$ formula where $x, y$ vary over elements of $BS(\omega_1)$. For an $x$ which appears in the iteration, in the course of the iteration we try to force $\varphi(x, y)$ to fail via a proper iteration of size $< \omega_2$ that does not add reals, for all $y$ that appear in the iteration; if this is not possible for some $y$ then we fix such a $y$ and we use "stationary-kill with localisation" methods to provide a $\Sigma_1$ witness for each $(x, y')$ with $y' \neq y$ (everything proper). Then $y$ is the unique $y$ such that $(x, y)$ is not witnessed, a $\Pi_1$ property. We also witness $(x, y')$ whenever we have forced $\varphi(x, y')$ to fail. The iteration is $\omega_2$-cc and does not add reals, giving the desired uniformisation.

# The Consistency of Uniformisation

The idea for $\Pi^1_3$ is the same, but now the proper iteration is of length $\omega_1$.

The argument also works for $BS(\kappa)$ for any successor cardinal $\kappa$. It also will likely work for inaccessibles which are not weakly compact. Obtaining $\Pi^1_1$ uniformisation for $BS(\kappa)$ for $\kappa$ weakly compact is an interesting question.

That's all, folks!