Undecidability of the the graph Homomorphism Problem for $\mathbb{Z}^2$ Actions

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A general question concerns the existence of continuous/Borel structurings for countable Borel equivalence relations.

- More generally, what is structure of the definable cardinals, and what kinds of structures exist on these objects?
- The class of countable equivalence relations provides a large source of examples of definable cardinals.
- Even when the underlying equivalence relation is fairly simple, the question about effective structurings of the quotient space X/E may be difficult.

The effective notion of cardinality comparison is the notion of a reduction of E on the space X to the relation F on the space Y. This means a map $f \colon X \to Y$ such that

$$(xEy) \Leftrightarrow (f(x)Ff(y))$$

This just says f is an injection from X/E to Y/F.

We can require that f be continuous, Borel, or arbitrary (in $\mathrm{ZF} + \mathrm{AD}$ contexts).

The context of Borel equivalence relations is a convenient way to present the theory of definable cardinalities, though sometimes the context matters.

Example

Woodin showed that assuming $AD_{\mathbb{R}}$ there are exactly five cardinalities below (including) ω_1^ω.

This is not true in all models of AD, however.

Recall (Feldman-Moore) that every countable Borel equivalence relation is the orbit equivalence relation of a Borel action of a countable group.

There is a natural action, the shift-action of the countable group G on the space 2^G given by

$$g \cdot x(h) = x(g^{-1}h)$$

This action is essentially universal for the actions of G, for example, any Borel action of G on X equivariantly embeds into the shift action of $G \times \mathbb{Z}$.

The definable cardinality of the orbit space X/E for E given by the shift action of G roughly corresponds to the algebraic complexity of G.

- If $G \leq H$ or $G = H/K$, then the shift action of G (equivariantly) embeds into the shift action of H.
- The same is true if we restrict to the free-part $F(2^G)$ of the shift action of G on 2^G.

For the simplest infinite group $G = \mathbb{Z}$, the Borel actions of G are all hyperfinite, that is, $E = \bigcup_n E_n$, an increasing union where each E_n is finite.

- All non-smooth hyperfinite relations are Borel bi-reducible, that is, the orbit spaces X/E have the same effective cardinality.
- By Harrinton-Kechris-Louveau this is the minimum cardinality above $\mathbb{R}$.

Question (Kechris, Weiss)

Is the Borel action of every amenable group hyperfinite?

Some results on the hyperfiniteness problem:

- All actions of $\mathbb{Z}^n$ are hyperfinite. (Weiss)
- All actions of a countable Abelian group are hyperfinite (Gao, J).
- All actions of a countable nilpotent group are hyperfinite (Seward, Schneider).
- There are actions of solvable, non-nilpotent groups of exponential growth with hyperfinite free actions (Conley, J, Marks, Seward, Tucker-Drob).

Though all these orbit spaces have the same effective cardinality, questions about effective structurings of these spaces are non-trivial, and may have different answers, even for the different $\mathbb{Z}^n$.

Many instances of effective (continuous/Borel) structuring problems can be phrased as sub-shift or graph homomorphism questions.

1.) A sub-shift of k^G is a closed, invariant $A \subseteq k^G$. A is of finite type if there is a finite set of $p_i \in k^{G_i}$ ($G_1 \subseteq G$ finite) such that $y \in A$ iff

$$\forall g\, (g \cdot y \restriction G_i \neq p_i).$$

2.) If $G = \langle G, S \rangle$ is a presentation of G, we have the Cayley graphing of $F(2^G)$. If Γ is a finite (or countable) graph, we can consider continuous/Borel graph homomorphisms from $F(2^G)$ to Γ.

A particular special case is when $G = \mathbb{Z}^n$.

- Although all abelian actions are hyperfinite the combinatorics remains interesting, and is connected with difficult questions about general marker structures in group actions (e.g., hyperfiniteness problem, union problem).
- Methods such as 2-colorings (hyperaperiodicity) and orthogonality are used both in sub-shift/graphing problems as well as hyperfiniteness arguments.

Some General Problems

Sub-shift problem: For which subshifts A of k^G does there exist a continuous/Borel equivariant map from $F(2^G)$ to A?

Graphing problem: Given $G = \langle G, S \rangle$, for which finite/countable graphs Γ does there exist a continuous/Borel graph homomorphism from $F(2^G)$ to Γ?

Tiling problem: Given finite sets ("tiles") $T_1, \dots, T_k \subseteq G$, does there exist a clopen/Borel set $M \subseteq F(2^G)$ such that $F(2^G) = \bigcup_{g \in m} T(g)$, where $T(g) \in \{T_1, \dots, T_k\}$.

An instance of the graphing problem is the chromatic number problem: determine the continuous/Borel chromatic number of $F(2^G)$.

Let $G = \mathbb{Z}^d$.

The hyperaperiodic/2-coloring theory produces a set of finite $\mathbb{Z}^2$-graphs $\Gamma_{n,p,q}$ which reduce the question of the existence of a continuous, equivariant map from $F(2^G)$ to $A = A(p_i)$ to finding such a map on some $\Gamma_{n,p,q}$.

This gives the following.

Theorem

The sub-shift problem is Σ^0_1, and thus so are the graph homomorphism and tiling problems.

We previously showed the following.

Theorem

The sub-shift problem is Σ^0_1*-complete.*

Here we show:

Theorem

The (continuous) graph homomorphism problem (for finite graphs) is Σ^0_1*-complete.*

Question

Is the continuous tiling problem for $2^{\mathbb{Z}^2}$ also Σ^0_1-complete?

Review of Hyperaperiodicity

For G a countable group, $x \in 2^G$ is a hyperaperiodic point (or a 2-coloring) if $\forall s \neq 1_G \; \exists T \in G^{<\omega}$ such that

$$\forall g \in G \; \exists t \in T \; (x(gt) \neq x(gst))$$

Fact

$x \in 2^G$ is hyperaperiodic iff $\overline{[x]} \subseteq F(2^G)$.

Fact (GJS)

Every countable group has a hyperaperiodic point.

Construction of $\Gamma_{n,p,q}$

- $\Gamma_{n,p,q}$ is constructed from 12 rectangular graphs each isomorphic to a rectangle region in $\mathbb{Z}^2$.
- Each has certain regions which are labelled. Vertices of the same label in the different tiles are identified.
- There are 4 torus tiles, 4 commutativity tiles, and 4 long tiles.

Torus tiles

$R_\times$	R_c	$R_\times$
R_a		R_a
$R_\times$	R_c	$R_\times$

$T_{ca=ac}$

$R_\times$	R_c	$R_\times$
R_b		R_b
$R_\times$	R_c	$R_\times$

$T_{cb=bc}$

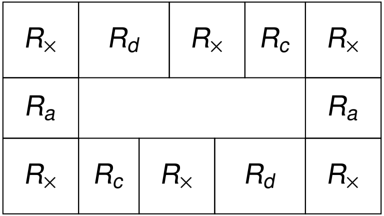
Plus $T_{da=ad}$ and $T_{db=bd}$.

$$R_x : n \times n$$

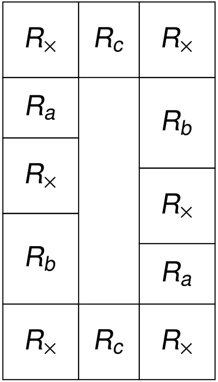
$$R_a : n \times (p - n) \qquad R_b : n \times (q - n)$$

$$R_c : (p - n) \times n \qquad R_d : (q - n) \times n$$

Commutativity tiles



$T_{dca=acd}$

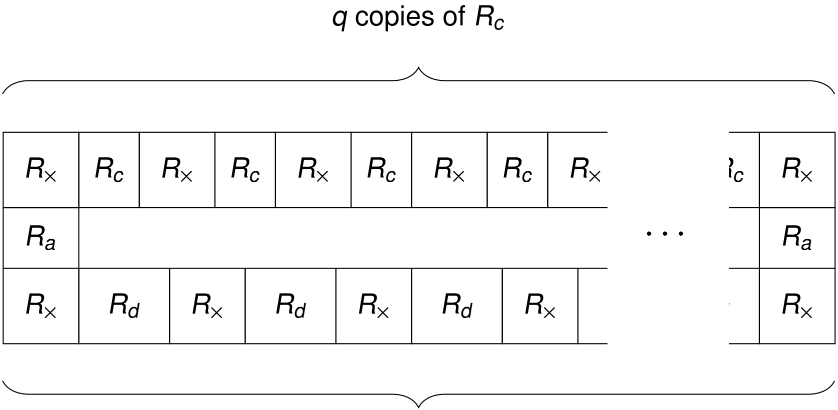


$T_{cba=abc}$

Plus $T_{cda=adc}$ and $T_{cab=bac}$.

Long tiles

$T_{c^q a = a d^p}$ (plus $T_{d^p a = a c^q}$, $T_{c b^p = a^q c}$, $T_{c a^q = b^p c}$).



Tiles Theorem for $\mathbb{Z}^2$

Theorem

Let $A \subseteq k^{\mathbb{Z}^2}$ be a subshift of finite type coded by $(p_1, \dots, p_i)$. Then the following are equivalent.

1. *There is a continuous, equivariant map $f \colon F(2^{\mathbb{Z}^2}) \to A$.*
2. *There is an n, p, q with $n < p, q$, $(p, q) = 1$, and $n \geq \max\{a_i, b_i \colon dom(p_i) = [0, a_i) \times [0, b_i)\}$ and a $g \colon \Gamma_{n,p,q} \to k$ which respects A.*
3. *For all $n \geq \max\{a_i, b_i \colon dom(p_i) = [0, a_i) \times [0, b_i)\}$, for all sufficiently large p, q with $(p, q) = 1$ there is a $g \colon \Gamma_{n,p,q} \to k$ which respects A.*

Let $\pi_1^*(\Gamma) = \pi_1(\Gamma)/N$, where $\pi_1(\Gamma)$ is the homotopy group (with fixed base point) and N is the normal subgroup generated by the 4-cycles.

Using the tiles theorem we have the following.

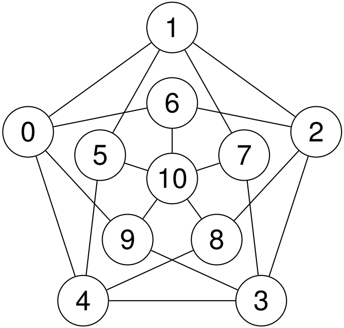
Theorem (Negative Condition)

Suppose $\forall N\, \exists p, q > N$ with $(p, q) = 1$ such that for every p-cycle γ in Γ, γ^q is not a pth power in $\pi_1^*(\Gamma)$. Then there is no continuous homomorphism from $F(2^{\mathbb{Z}^2})$ to Γ.

Theorem (Positive Condition)

Suppose there is an odd cycle $\gamma \in \Gamma$ which has finite order in $\pi_1^*(\Gamma)$. Then there is a continuous homomorphism from $F(2^{\mathbb{Z}^2})$ to Γ.

As a consequence, every graph Γ without 4-cycles satisfies the negative condition. Also, if $\chi(\Gamma) \leq 3$, there is no continuous homomorphism.



0	1	2	3	9	0
6	2	1	7	10	6
0	1	5	10	6	0
6	0	4	8	2	6
0	9	3	2	1	0

Figure: The Grötzsch Graph. The odd cycle $\gamma = (0, 1, 2, 3, 9, 0)$ has order 2.

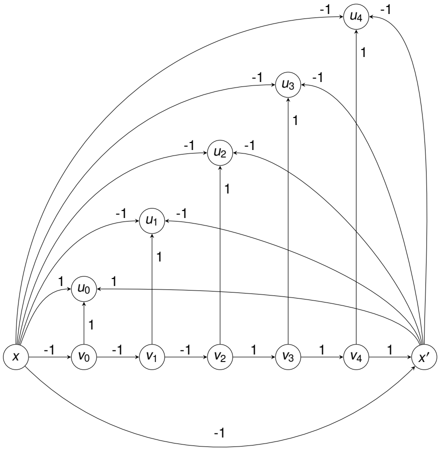


Figure: The “Clamshell” graph together with a weight function verifying the negative condition.

Theorem

The set of finite graphs Γ for which there is a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to Γ is Σ^0_1-complete.

We reduce a variation of the word problem for groups to the graph homomorphism problem, specifically, the word problem for torsion-free groups.

There is a Σ^0_1-complete set $C \subseteq \omega$ and a recursive map f such that $f(n)$ is the code of a presentation $\mathcal{G}_n$ of a finitely presented torsion-free group G_n

$$\mathcal{G}_n = \langle a_1, \ldots, a_k | r_1, \ldots, r_\ell \rangle$$

and a word $w = w_n = w_n(a_1, \ldots, a_k)$ such that

$$n \in C \quad \text{iff} \quad (w_n = 1 \text{ in } G_n)$$

We add the extra generator z and relation $z^2 = w$ to form G'_n. That is:

$$\mathcal{G}'_n = \langle a_1, \dots, a_k, z \,|r_1, \dots, r_\ell, z^2 w^{-1} \rangle$$

Let $f'(n)$ code this presentation.

We associate a graph Γ'_n to G'_n.

Lemma

Let $\mathcal{G} = \langle b_1, \dots, b_k | s_1, \dots, s_\ell \rangle$ be a finite presentation for a group G. Then there is a graph $\Gamma(\mathcal{G})$ such that $\pi_1^*(\Gamma(\mathcal{G})) \cong G$. Moreover, the map g given by $\mathcal{G} \mapsto g(\mathcal{G}) = \Gamma(\mathcal{G})$ is recursive.

Let $h = g \circ f'$. So, $h(n)$ codes $\Gamma(\mathcal{G}'_n)$.

We may assume that all of the generators for $f'(n)$ except z map to cycles of even length, while z maps to a cycle of odd length.

- If $n \in C$ then $z^2 = w = 1$ in $\mathcal{G}'_n$, and so $\Gamma(\mathcal{G}'_n)$ satisfies the positive condition.
- Suppose $n \notin C$, so w has infinite order in $\mathcal{G}_n$. We show that $\Gamma(\mathcal{G}'_n)$ satisifies the negative condition.

Recall $\pi_1^*(\Gamma(\mathcal{G}'_n)) \cong G'_n \cong G_n *_{H,K} \mathbb{Z}$, where $H = \langle w \rangle \leq G_n$, $K = \langle z^2 \rangle \leq \mathbb{Z}$. Let U be a set of non-identity coset representatives

for the subgroup H of G_n.

The normal form theorem for amalgamated free products says that every element v of G'_n can be written uniquely in one of the forms

$$v = gu_1zu_2z\cdots zu_n \quad (1)$$

$$v = gzu_1zu_2\cdots zu_n \quad (2)$$

$$v = gu_1zu_2z\cdots u_nz \quad (3)$$

$$v = gzu_1zu_2\cdots u_nz \quad (4)$$

$$v = gz \quad (5)$$

where $u_i \in U$ and $g \in H$.

To each $x \in G'_n$ we assign an integer $i(x) \in \mathbb{N}$ defined as follows.

Definition

Let $x \in G'_n$. Among all representations of x as a product of the form $x = hvh^{-1}$ where v is in normal form, we let $i(x)$ be the minimum number of occurrences of z in the normal form v.

If x is odd we have that $i(x) > 0$.

Lemma

Let $x \in G'_n$. Write $x = hvh^{-1}$ where v is in normal form, not in case (5), and the number of occurrences of z in v is equal to $i(x)$. Then for any $m > 0$ we have that $x^m = hv^m h^{-1}$ where the normal form for v^m has $m \cdot i(x)$ many z's.

Let p, q be large odd primes.

Suppose γ is a p-cycle in $\Gamma(G'_n)$, γ^q is a p power in $\pi_1^*(\Gamma(G'_n))$.

Say $\gamma^q = \delta^p$ in $\Gamma(G'_n)$.

In G'_n we may write $\gamma = hvh^{-1}$, $\delta = kuk^{-1}$, where $i(\gamma)$, $i(\delta)$ are attained by v, u.

We consider the case u, v not in case (5).

$$hv^q h^{-1} = ku^p k^{-1}$$

v^q, u^p are in reduced forms. We have $v^{qN} = h^{-1}ku^{pN}k^{-1}h$ for any N.

We must have $p|i_v$ as otherwise $|(i_v qN - i_u pN)| \geq N$, a contradiction for large enough N.

Recall γ is a p-cycle in $\Gamma(G'_n)$.

There is a fixed small constant r such that $\gamma = \gamma'$ in $\pi_1^*(\Gamma(G'_n))$ of length $|\gamma'| \leq r|\gamma|$, where γ' is a word in the generators of G'_n.

There are at least $i_v \geq p$ many z's in the reduced form of γ'. So, $|\gamma'| \geq p|z|$. Since we may assume $|z| > r$, this is a contradiction.