Absolute $F_{\sigma\delta}$ spaces

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Motivation

All our spaces will be Tychonoff.

Absoluteness of low descriptive classes

Let X be a topological space and cX its compactification. Then

1. X is *open* in $cX \iff X$ is locally compact,
2. X is G_δ in $cX \iff X$ is Čech-complete,
3. X is *closed* in $cX \iff X$ is compact,
4. X is F_σ in $cX \iff X$ is σ-compact.

Motivation

All our spaces will be Tychonoff.

Absoluteness of low descriptive classes

Let X be a topological space and cX its compactification. Then

1. X is *open* in cX $\iff$ X is locally compact $\iff$ X is open in every compactification,
2. X is G_δ in cX $\iff$ X is Čech-complete $\iff$ X is G_δ in every compactification,
3. X is *closed* in cX $\iff$ X is compact $\iff$ X is closed in every compactification,
4. X is F_σ in cX $\iff$ X is σ-compact $\iff$ X is F_σ in every compactification.

In other words, every closed space (:= closed in some compactification) is absolutely closed (:= closed in every compactification). Analogously for 'open', 'G_δ' and 'F_σ'.

‘Problems’ with higher classes

An obvious conjecture: the same holds for all descriptive classes. However...

Example (Talagrand, 1985)

There exists an $F_{\sigma\delta}$ space which is not absolutely $F_{\sigma\delta}$.

My topics of interest:

1. When is an $F_{\sigma\delta}$ space absolutely $F_{\sigma\delta}$? (first part of the talk)
2. Complexity vs absolute complexity - which combinations are possible

$$\text{(in ‘}X\text{ is of class }\Gamma\text{ in }cX\text{, but of class }\Psi\text{ in }dX\text{’)?}$$

(second part of the talk)
3. ...and more questions (to which I don’t know the answer to yet).

Towards the sufficient condition: complete sequences of covers

Definition

A sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of covers of X is said to be *complete*, if for every filter $\mathcal{F}$ on X, we have

$$(\forall n \in \mathbb{N})(\mathcal{F} \cap \mathcal{C}_n \neq \emptyset) \implies (\exists x \in X)(\forall U \in \mathcal{U}(x))(\forall F \in \mathcal{F}) : U \cap \overline{F} \neq \emptyset.$$

This notion is connected to descriptive complexity in the following way:

Theorem (Frolík)

1. X is Čech-complete $\iff$ X has a complete sequence of open covers.
2. X is $\mathcal{K}$-analytic $\iff$ X has a complete sequence of countable covers.
3. X is $F_{\sigma\delta}$ $\iff$ X has a complete sequence of countable closed covers.

Sufficient condition for a space to be absolutely $F_{\sigma\delta}$

Theorem (Frolík)

1. X is Čech-complete $\iff$ X has a complete sequence of open covers.
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Problematic question number one (Frolík)

Describe those topological spaces which are $F_{\sigma\delta}$ in every compactification.

Theorem 1 (Kalenda, K.)

X is absolutely $F_{\sigma\delta}$ $\Leftarrow$ X has a compl. seq. of countable *disjoint* F_σ covers.

We can get away with less - but it is not a characterization (yet, anyway).

Consequences

Definition

A topological space X is said to be hereditarily Lindelöf if every open cover of every subspace Y of X has a countable sub-cover.

(in particular, separable metrizable spaces are hereditarily Lindelöf)

Corollary

A hereditarily Lindelöf space which is $F_{\sigma\delta}$ is absolutely $F_{\sigma\delta}$.

Proposition (Holický, Spurný)

For hereditarily Lindelöf spaces, ($\mathcal{F}$-Borel) complexity is automatically absolute.

Corollary

Every separable Banach space is absolutely $F_{\sigma\delta}$ (when equipped with weak topology).

Second part: $\mathcal{F}$-Borel classes

We will need the following definition:

Definition ($\mathcal{F}$-Borel sets)

We denote

- $\mathcal{F}_1(X) :=$ closed subsets of X,
- $\mathcal{F}_2(X) := F_\sigma$ subsets of X,
- $\mathcal{F}_3(X) := F_{\sigma\delta}$ subsets of X,
- $\mathcal{F}_\alpha(X) := \left(\bigcup_{\beta<\alpha} \mathcal{F}_\beta(X) \right)_\sigma$ for $1 < \alpha < \omega_1$ even,
- $\mathcal{F}_\alpha(X) := \left(\bigcup_{\beta<\alpha} \mathcal{F}_\beta(X) \right)_\delta$ for $1 < \alpha < \omega_1$ odd.

Talagrand's broom spaces

Talagrand has constructed an $F_{\sigma\delta}$ space X, such that for one of its compactifications cX, X does not belong to any of the classes $\mathcal{F}_\alpha(cX)$, $\alpha < \omega_1$.

Based on his construction, we have obtained the following result:

Talagrand's examples and their properties

For every two countable ordinals $\alpha \geq \beta \geq 3$, α odd, there exists a space X^α_β, such that

1. the complexity of X^α_β is $\mathcal{F}_\beta$;
2. the absolute complexity of X^α_β is $\mathcal{F}_\alpha$.

Notes:

- By 'complexity' we mean that it belongs to the given class, but not to any lower class.
- The lower bound on the absolute complexity is Talagrand's.

Canonical $\mathcal{F}_\alpha$-sets containing X

Every 'nice' (at least $\mathcal{K}$-analytic) space has a monotone Suslin scheme, that is, a sequence of covers $(\mathcal{C}_n)_n$ satisfying:

- $\mathcal{C}_n = \{C_s |\ s \in \mathbb{N}^n\}$;
- For each sequence s: $C_s = \bigcup\{C_{s\hat{}k} |\ k \in \mathbb{N}\}$.

The sets X_n for $n = 0, 1, 2, \ldots$

In every compactification cX, we define:

- $X_0 := \overline{X}^{cX}$
- $X_1 := \bigcap_n \bigcup_{s \in \mathbb{N}^n} \overline{C_s}^{cX}$
- $X_2 := \bigcap_n \bigcup_{s \in \mathbb{N}^n} \bigcap_n \bigcup_{t \in \mathbb{N}^k} \overline{C_{s\hat{}t}}^{cX}$
- And so on for any $\alpha < \omega_1$. 'Huh, what about $\alpha \geq \omega$?'

Admissible mappings and X_α

$S_\alpha :=$ ω-ary tree of height α (on $\mathbb{N}$)
(Vojta, dont be lazy and draw this)

Definition

A mapping $\varphi : S_\alpha \to$ (finite sequences on $\mathbb{N}$) is *admissible* if it satisfies:

1. $\forall s = (s_1, s_2, \dots, s_n)$: the length of $\varphi(s)$ is $s_1 + s_2 + \cdots + s_n$;
2. $\forall s, t : s$ extends $t \implies \varphi(s)$ extends $\varphi(t)$.

Definition of X_α

Let cX be a fixed compactification. For every $\alpha < \omega_1$, we define

$$X_\alpha := \{x \in cX |\ \exists \varphi : S_\alpha \to \mathcal{S} \text{ admissible s.t. } \forall s \in S_\alpha : x \in \overline{C_{\varphi(s)}}^{cX}\}.$$

To prove the main result, we show that for Talagrand's broom spaces, $X = X_\alpha$ holds for a suitable α (and α does not depend on the chosen compactification).

Problematic question number two

Question 2

Are the 'nice $F_{\sigma\delta}$ Banach spaces' (:= $F_{\sigma\delta}$ in second dual) absolutely $F_{\sigma\delta}$?

Definition

By $c_0(\Gamma)$ we denote the space of long sequences with limit 0:

$$c_0(\Gamma) := \{f \in \mathbb{R}^\Gamma \,|\, \forall \epsilon > 0 : |f(\gamma)| \geq \epsilon \text{ only holds for finitely many } \gamma \in \Gamma\}.$$

$c_0(\Gamma)$ is an example, in some sense canonical, of a nice $F_{\sigma\delta}$ Banach space - but is it absolutely $F_{\sigma\delta}$?
A complication: method used for Talagrand's broom spaces only works for 'simple' spaces, it cannot be applied here.

The End

Thank you for your attention!