Simple witnesses of Haar null sets

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Small sets

It is common in mathematics to find properties that are true for 'almost all' points of a space, but false for a *small set* of exceptional points.

A notion of smallness is 'good' if:

- the whole space is not small (and the empty set is small)
- the system of small sets forms a σ-ideal, that is
  - the union of countably many small sets is small
  - a subset of a small set is small
- a *translate* of a small set is small

Definition

In a group $(G, \cdot)$ the translates of a set $N \subseteq G$ are the sets of the form $gNh = \{gnh : n \in N\}$ where $g, h \in G$ are fixed elements.

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Haar null sets

In locally compact groups sets of Haar measure zero provide a good notion of smallness (meager sets are also a good choice, but with a very different 'meaning').

In 1972 Christensen noticed that although Haar measures exist only in locally compact groups, 'sets of Haar measure zero' can be generalized to all Polish groups:

Definition (Haar null set)

In a Polish group, a Borel set N is called *Haar null* iff there exists a Borel probability measure which assigns measure zero to all translates of N. A probability measure satisfying this condition is called a *witness measure* (for the set N).

Remarks: Haar null sets are also called *shy sets*. A non-Borel set is Haar null iff it is the subset of a Borel Haar null set.

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In a Polish group G the system of Haar null sets is a translation-invariant σ-ideal. G itself is not a Haar null set.

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Motivation

Theorem (well-known)

Every set of Lebesgue measure zero (for example in $\mathbb{R}$) is the subset of a G_δ set of Lebesgue measure zero.

Theorem (Elekes, Vidnyánszky)

If G is a non-locally-compact abelian Polish group, then there is a Haar null set in G that is not a subset of any G_δ Haar null set.

Theorem (D.N.)

There is a $F_{\sigma\delta}$ Haar null set in $\mathbb{Z}^\omega$ that is not a subset of any G_δ Haar null set.

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Existence of simple witnesses

Definition

Let $x \in \mathbb{Z}_+^\omega$ be a fixed sequence of positive integers. For all $i \in \omega$ choose a random integer $z_i \in \{0, 1, 2, \ldots, x_i\}$ uniformly and independently of the choice of the other z_j's. This procedure defines a probability measure μ_x on $\mathbb{Z}^\omega$. We will call measures of this form *product-uniform measures* (because they are products of uniform measures).

Theorem (D.N.)

If $N \subset \mathbb{Z}^\omega$ is Haar null, it has a product-uniform witness measure.

This result is motivated by a similar result of Solecki which proves that every Haar null set has a 'simple' witness measure (using another similar notion of 'simple' measures).

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Modifying witness measures

In the proof we will take an arbitrary witness measure and 'modify' it to get simpler witness measures. Suppose that in a Polish group G we have a Haar null set N with a witness measure μ. Then:

Restriction (and renormalization)

If $P \subseteq G$ is a Borel set with $\mu(P) > 0$ then $\mu_P(X) = \mu(X \cap P)/\mu(P)$ is also a witness measure for N.

Convolution

If ν is another Borel probability measure then the convolution $\mu * \nu$ defined by $(\mu * \nu)(X) = (\mu \times \nu)(\{(u, v) \in G \times G : u \cdot v \in X\})$ is also a witness measure for N.

Translation (from the left)

If $g \in G$ then the measure $g \cdot \mu$ defined by $(g \cdot \mu)(X) = \mu(g^{-1}X)$ is also a witness measure for N.

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Proof: $(\mu * \nu)(gNh) = \int_G \underbrace{\mu(gNhv^{-1})}_{=0} \, \mathrm{d}\nu(v) = 0$ $(\forall g, h \in G)$.

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Compact support

Let $N \subset \mathbb{Z}^\omega$ be a Haar null set. It is known that in a Polish space Borel measures are *tight*, that is, the measure of a measurable set is the supremum of the measures of its compact subsets.

$\Downarrow$ Restriction

We can modify an arbitrary witness measure of N to get a witness measure which has a compact support.

If $K \subset \mathbb{Z}^\omega$ is compact, then as the projection functions are continuous, we get that for each $i \in \omega$ the set $\{z_i : z \in K\} \subset \mathbb{Z}$ is compact (i.e. finite).

$\Downarrow$ Translation (if necessary)

Claim

There exist positive integers m_i ($i \in \omega$) and a measure μ which is a witness measure for N, such that $-m_i \leq z_i \leq 0$ for all $z \in \operatorname{supp} \mu$.

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Notice that we chosen a μ such that $\operatorname{supp} \mu$ consists of sequences with *non-positive* elements; this will be useful later in the proof.

Now we are ready to do the core step of the proof: we apply Convolution and then Restriction to get a product-uniform measure.

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Detour: One-dimensional analog

Let m be a positive integer and consider an arbitrary probability measure $\mu^{(1)}$ on $\mathbb{Z}$ such that

$$\operatorname{supp} \mu^{(1)} \subseteq \{-m, -m+1, \ldots, -1, 0\} \subset \mathbb{Z}.$$

Let $p_i = \mu^{(1)}(\{i\}) \in [0, 1]$, then by definition $p_{-m} + p_{-m+1} + \ldots + p_1 + p_0 = 1$ and all other p_i's are zeroes.

Let $n \gg m$ be a large integer and let $\nu^{(1)}$ be the uniform probability measure with support $\{0, 1, \ldots, n\} \subset \mathbb{Z}$.

Notice that the convolution $\mu^{(1)} * \nu^{(1)}$ assigns measure $(p_{-m} + \ldots + p_0)/(n+1) = 1/(n+1)$ to each $j \in \{0, 1, \ldots n - m\}$.

We can restrict and rescale this convolution to get a uniform probability measure with support $\{0, 1, \ldots, n - m\}$.

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Back to the original problem

We have a Borel probability measure μ on $\mathbb{Z}^\omega$ such that (it is a witness measure for N and) there exist positive integers m_i ($i \in \omega$) such that $-m_i \leq z_i \leq 0$ for all $z \in \operatorname{supp} \mu$.

Choose a product-uniform measure $\nu = \nu_n$ where the sequence $n \in \mathbb{Z}_+^\omega$ satisfies that $n_i \gg m_i$ and $\prod_{i \in \omega}((n_i - m_i)/(n_i + 1)) > 0$.

$\Downarrow$ Convolution with ν

If S satisfies that $0 \leq h_i \leq n_i - m_i$ for every $h \in H$ and $i \in \omega$ then $(\mu * \nu)(S) = \int_{\operatorname{supp} \mu} \underbrace{\nu(-u + S)}_{=\nu(S)} \, \mathrm{d}\mu(u) = \mu(S)$.

$\Downarrow$ Restriction to the set $\{z \in \mathbb{Z}^\omega : 0 \leq z_i \leq n_i - m_i \text{ for all } i \in \omega\}$

The product-uniform measure $\nu_{(n-m)}$ is a witness measure. □

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Generalizations

Theorem (D.N.)

If $N \subset \mathbb{Z}^\omega$ is Haar null, it has a product-uniform witness measure.

Question (1)

If $G = \prod_{i\in\omega} G_i$ where the G_i's are countable groups and all but finitely many of them are amenable, then does every Haar null set $N \subset G$ have a product-uniform witness measure (for some natural generalization of product-uniform)?

I think this is true; the related results of Solecki were proved in this class of groups.

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Question (2)

If $G = \prod_{i\in\omega} G_i$ where the G_i's are countable groups, then does every Haar null set $N \subset G$ have a product-uniform witness measure?