A remark on Glasner's problem

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Veech's problem

Definition

Let G be a group.

- *A character is a group homomorphism $\chi : G \to S^1$.*
- *A subset $S \subset G$ is syndetic when $G = FS$ for some finite $F \subset G$.*

Theorem (Veech, 68)

Let $S \subset \mathbb{Z}$ be syndetic. Then for some characters $\chi_1, ..., \chi_n$ on $\mathbb{Z}$, $\varepsilon > 0$

$$\{n \in \mathbb{Z} : \forall i \leq n \quad |\chi_i(n) - 1| < \varepsilon\} \subset^* S - S$$

where $\subset^*$ means "up to a Banach density zero subset".

Remark

This improves on former results of Følner, who proved similar versions for

- *$... \subset S - S + S - S + S - S + S - S$ (Følner, 46)*
- *$... \subset S - S + S - S$ (Følner, 54)*

Veech's problem, continued

Question (Veech, 68)

Let $S \subset \mathbb{Z}$ be syndetic. Are there characters $\chi_1, ..., \chi_m$ on $\mathbb{Z}$, $\varepsilon > 0$

$$\{n \in \mathbb{Z} : \forall j \leq m \quad |\chi_j(n) - 1| < \varepsilon\} \subset S - S?$$

Glasner observed that the existence of a specific kind of Polish group would provide a negative answer. More precisely:

Definition

Let G be a topological group.

- *G is monothetic when it admits a dense cyclic subgroup.*
- *G is extremely amenable when every G-flow has a fixed point. (A G-flow is a continuous action of G on a compact space X. Notation: $G \curvearrowright X$.)*
- *G is minimally almost periodic when the only continuous character on G is the constant one.*

Glasner's problem, continued

Theorem (Glasner, 98)

Assume that there exists an infinite monothetic, minimally almost periodic, Polish group that is not extremely amenable.
Then Veech's problem has a negative answer.

This naturally leads to:

Question ("Glasner's problem", 98)

Is there an infinite monothetic, minimally almost periodic, Polish group that is not extremely amenable?

Remark

Glasner's opinion is that such groups do exist.

Recasting the question: Universal minimal flows

Definition

Let G be a Polish group, and $G \curvearrowright X$ a G-flow.

- *$G \curvearrowright X$ is minimal when the orbit of every $x \in X$ is dense.*
- *$G \curvearrowright X$ is universal for minimal G-flows when every minimal G-flow is a factor of $G \curvearrowright X$:*
  If $G \curvearrowright Y$ minimal, there is $\pi : X \twoheadrightarrow Y$ continuous and equivariant.

$$\forall g \in G \quad \forall x \in X \quad \pi(g \cdot x) = g \cdot \pi(x).$$

Theorem (Ellis)

Let G be a topological group. Then G admits a universal minimal G-flow (UMF), and it is unique. Notation: $G \curvearrowright M(G)$.

Facts

$M(G)$ is a compact topological space that can be:

- trivial. This is the same as saying that G is extremely amenable.
- non-trivial, but metrizable (eg: G compact; $Homeo_+(S^1)$, Pestov 98; $\mathrm{Aut}(\mathbb{F})$ for certain Fraïssé $\mathbb{F}$, Kechris-Pestov-Todorcevic 05).
- non-metrizable (eg: G locally cpct, Kechris-Pestov-Todorcevic 05).

Glasner's problem therefore asks whether every monothetic, minimally almost periodic, Polish group can have a non-trivial UMF.

Theorem (NVT)

Let G be a monothetic, minimally almost periodic, Polish group. If $M(G)$ is metrizable, then it is trivial (ie G is extremely amenable).

Remark

Ben Yaacov-Melleray-Tsankov also have a (slightly different) proof of this.

Recasting the question: more flows

Definition

Let $G \curvearrowright X$ be a G-flow. An ordered pair $(x, y) \in X^2$ is:

- *proximal when $g \cdot x$ and $g \cdot y$ can be made arbitrarily close.*
- *distal when it is not proximal.*

Definition

A G-flow $G \curvearrowright X$ is:

- *proximal when every $(x, y) \in X^2$ is proximal.*
- *distal when every $(x, y) \in X^2$ with $x \neq y$ is distal.*
- *equicontinuous when*

$$\forall U \in Unif(X) \;\; \exists V \in Unif(X) \;\; \forall x, y \in X$$

$$(x, y) \in V \Rightarrow \forall g \in G \;\; (g \cdot x, g \cdot y) \in U$$

Universal minimal flows and fixed-points properties

Theorem

Let G be a Polish group. Then each of the previous classes of flows admits a unique universal minimal flow. Notation: $G \curvearrowright \Pi(G)$ for proximal UMF, $G \curvearrowright D(G)$ for distal UMF, $G \curvearrowright B(G)$ for equicontinuous UMF.

Definition

Let G be a topological group. It is strongly amenable when every proximal G-flow has a fixed point (equiv. $\Pi(G)$ trivial).

Remark

Every abelian (hence monothetic) topological group is strongly amenable.

Theorem

Let G be a topological group. Then TFAE:

- *every equicontinuous G-flow has a fixed point (equiv. $B(G)$ trivial),*
- *every distal G-flow has a fixed point (equiv. $D(G)$ trivial),*
- *G is minimally almost periodic.*

Ingredients for the proof

By results from Melleray-NVT-Tsankov and Ben Yaacov-Melleray-Tsankov:

Theorem

Let G be a Polish group with $M(G)$ metrizable. Then:

1. *There is $G^* \leq G$, closed, co-precompact, extremely amenable, such that $M(G) = \widehat{G/G^*}$.*
2. *There is $G^{**} \leq G$, closed, co-precompact, strongly amenable, such that $\Pi(G) = \widehat{G/G^{**}}$, namely $G^{**} = N(G^*)$ (normalizer of G^* in G).*

In particular, G is strongly amenable iff G^* is normal in G.

Theorem (NVT)

Let G be a Polish group with $M(G)$ metrizable, and G^* as above. Then $B(G) = G/(G^*)^G$, where $(G^*)^G$ denotes the normal closure of G^* in G.

Proof of the main theorem

Theorem (NVT)

Let G be a strongly amenable, minimally almost periodic, Polish group. If $M(G)$ is metrizable, then it is trivial, ie G is extremely amenable.

Proof.

By the previous results:

- By metrizability of $M(G)$, there is $G^* \leq G$, closed, co-precompact, extremely amenable, such that $M(G) = \widehat{G/G^*}$.
- By strong amenability, G^* is normal in G, so $(G^*)^G = G^*$.
- By minimal almost periodicity, $(G^*)^G = G$.

It follows that $G^* = G$, ie G is extremely amenable.

□

Glasner's problem revisited

In view of the previous results, here is a variation of Glasner's problem:

Question

Let G be a strongly amenable, minimally almost periodic, Polish group. Is G extremely amenable?