Cardinal invariants of the Haar null ideal

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Introduction

The Haar null ideal $\mathcal{HN}(G)$.

Definition

Let G be a Polish group, $N \subseteq G$ be a set. Then N is Haar null, $N \in \mathcal{HN}(G)$ if there exists a Borel set $B \supseteq N$, and a probability Borel measure μ on G such that

$$\forall g, h \in G \quad \mu(hBg) = 0.$$

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The original notion $\mathcal{HN}_{\mathcal{UM}}(G)$, in the sense of Christensen:

Definition

Let G be a Polish group, $N \subseteq G$ be a set. Then N is generalized Haar null, $N \in \mathcal{HN}_{\mathcal{UM}}(G)$ if there exists a universally measurable set $U \supseteq N$, and a probability Borel measure μ on G such that

$$\forall g, h \in G \quad \mu(hUg) = 0.$$

Facts

It is a generalisation of the null ideal:

Theorem

If G is a locally compact Polish group, then $\mathcal{HN}(G) = \mathcal{HN}_{\mathcal{UM}}(G)$, moreover it is the set of null sets w.r.t. the (complete) left or right Haar measure.

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Theorem (M. Elekes, Z. Vidnyánszky)

Suppose that G is a non-locally compact Polish group that admits an invariant metric. Then there is a set $C \in \mathcal{HN}_{\mathcal{UM}}(G) \cap \mathbf{\Pi}^1_1$ such that

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i.e. $\exists \mu$ probability Borel measure s.t. $\mu^*(gCh) = 0$ $(\forall g, h)$,
$\nexists\ \nu, B \in \mathbf{\Delta}^1_1$ such that $C \subseteq B$, and $\nu(gBh) = 0$ $(\forall g, h)$.

A sufficient condition for non-Haar nullness

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In a topological group G a set S is called compact-catcher, if for every compact set $C \subseteq G$ there exist elements g, h such that

$$gCh \subseteq S \quad (\iff \quad C \subseteq g^{-1}Sh^{-1})$$

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Then translating B so that it covers C

$$C \subseteq gBh \ \Rightarrow \ 0 < \mu(C) \leq \mu(gBh).$$

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A Haar null set $B \in \mathbf{\Delta}^1_1$ does not necessarily have a G_δ hull, in fact the following holds.

Theorem (M. Elekes, Z. Vidnyánszky)

Suppose that G is a non-locally compact Polish group with an invariant metric, and let $\alpha < \omega_1$. Then there exists a Haar null Borel set $B_\alpha \in \mathcal{HN}(G)$ such that there is no $B' \in \mathcal{HN}(G)$ with $B_\alpha \subseteq B'$ and $B' \in \mathbf{\Pi}^0_\alpha$.

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If G is non-locally compact with an invariant metric then

$$\mathrm{add}(\mathcal{HN}(G)) = \omega_1.$$

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Corollary

If G is non-locally compact with an invariant metric then

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Proof. Let B_α-s ($\alpha < \omega_1$) be given by the theorem. Now if $B \supseteq \bigcup_{\alpha<\omega_1} B_\alpha$ is a Borel Haar null set, and $B \in \mathbf{\Pi}^0_\beta$ for some countable β, $B_\beta \subseteq B$ is a contradiction.

Uniformity and covering number

Theorem (T. Banakh)

$$\mathrm{cov}(\mathcal{HN}_{\mathcal{UM}}(\mathbb{Z}^\omega)) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{N}))$$

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Cofinality

Theorem (T. Banakh)

Let G be a non-locally compact Polish group admitting an invariant metric. Then

$$\mathrm{cof}(\mathcal{HN}_{\mathcal{UM}}(G)) > \min(\mathfrak{d}, \mathrm{non}(\mathcal{N})).$$

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Corollary

(MA) Let G be a non-locally compact Polish group admitting an invariant metric. Then

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The proof consists of the following parts. Recall that for a Polish space X the Effros Borel space $\mathcal{F}(X)$ is the standard Borel space of the closed subsets of X. The essence of the proof lies in the following technical statement.

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Proposition

Let G be a non-locally compact Polish group with an invariant metric. Then there exists a family of closed Haar null sets $\{N_x : x \in 2^\omega\} \subseteq \mathcal{HN}(G) \cap \mathcal{F}(G)$ *such that*

the mapping $x \mapsto N_x$ *is an injective Borel mapping from* 2^ω *to* $\mathcal{F}(G)$,

and for each nonempty perfect set $P \subseteq 2^\omega$

$$\bigcup_{x \in P} N_x \text{ is compact-catcher.}$$

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Proof.(Theorem)
For $\mathrm{cof}(\mathcal{HN}(G)) = \mathfrak{c}$ it is enough to show that for each Borel set $B \in \mathcal{HN}(G)$

$$|\{x \in 2^\omega : \ N_x \subseteq B\}| \leq \omega_1.$$

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Then, since $\{(g, F) \in G \times \mathcal{F}(G) : g \in F\} \subseteq G \times \mathcal{F}(G)$ is Borel,

$$\{F \in \mathcal{F}(G) : F \subseteq B\} = \{F : \forall g \; g \in F \rightarrow g \in B\}$$

is $\mathbf{\Pi}^1_1$,

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is $\mathbf{\Pi}^1_1$, so is its intersection with $\{N_x : x \in 2^\omega\}$:

$$\{N_x \in \mathcal{F}(G) : N_x \subseteq B\} \in \mathbf{\Pi}^1_1$$

($\{N_x : x \in 2^\omega\}$ is Borel, since $x \mapsto N_x$ was injective and Borel).

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($\{N_x : x \in 2^\omega\}$ is Borel, since $x \mapsto N_x$ was injective and Borel).
Now taking its preimage under the mapping $x \mapsto N_x$

$$\{x \in 2^\omega : \ N_x \subseteq B\}$$

is also $\mathbf{\Pi}^1_1$, thus it is the union of ω_1-many Borel sets.

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$$D \subseteq \{x \in 2^\omega : \ N_x \subseteq B\} \subseteq 2^\omega,$$

which contains a nonempty perfect set $P \subseteq D$.

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which contains a nonempty perfect set $P \subseteq D$.

But $\bigcup_{x \in P} N_x$ is compact-catcher, $\bigcup_{x \in P} N_x \subseteq B$ is Haar null, a contradiction.

Question

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Can the results $\mathrm{cov}(\mathcal{HN}(\mathbb{Z}^\omega)) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{N}))$ *and* $\mathrm{non}(\mathcal{HN}(\mathbb{Z}^\omega)) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{N}))$ *be generalised to non-locally compact Polish groups*

- *with invariant metric?*
- *which are Abelian?*
- *or at least for Banach spaces?*

Thank you for your attention!