Combinatorial Variants of Lebesgue's Density Theorem

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Lebesgue's Density Theorem

Definition

Suppose that (X, d, μ) is a Polish metric space with a Borel measure μ. An element x of X is a *μ-density point* of a subset A of X if

$$\liminf_{\epsilon > 0} \frac{\mu(B_\epsilon(x) \cap A))}{\mu(B_\epsilon(x))} = 1.$$

Theorem

(1) *(Lebesgue) Suppose that A is a Lebesgue measurable subset of $\mathbb{R}^n$ with the Lebesgue measure and $D_{\mathrm{L}}(A)$ is the set of Lebesgue density points of A. Then $\mu(A \triangle D_{\mathrm{L}}(A)) = 0$.*

(2) *(Miller) Suppose that (X, d, μ) is an ultrametric Polish space with a finite Borel measure μ and $A \subseteq X$ is μ-measurable. Let $D_{\mathrm{L},\mu}(A)$ be the set of μ-density points of A. Then $\mu(A \triangle D_{\mathrm{L},\mu}(A)) = 0$.*

Lebesgue's Density Theorem

Theorem (Vitali's Covering Theorem)

Given a collection of open balls centered at the points of a set $A \subseteq \mathbb{R}^n$ that contains arbitrary small balls at each point in A, there is a disjoint subcollection that covers A except for a null set.

Proof of Lebesgue's Density Theorem.

- We claim that $A_\epsilon = \{x \in \mathbb{R}^n \mid \limsup_{r \to 0} \frac{\mu(B_r(x) \setminus A)}{\mu(B_r(x))} > \epsilon\}$ is a null set for all $\epsilon > 0$.
- For any $\delta > 0$, let U_δ be an open set with $A_\epsilon \subseteq U_\delta$ and $\mu(U_\delta \setminus A_\epsilon) < \delta$. We obtain a collection $\mathcal{C}$ from Vitali's Covering Theorem for the collection of all open balls $B \subseteq U_\delta$ with $\frac{\mu(B \setminus A)}{\mu(B)} > \epsilon$ at elements of A_ϵ. Then

$$\epsilon \mu(A_\epsilon) \leq \epsilon \sum_{B \in \mathcal{C}} \mu(B) < \sum_{B \in \mathcal{C}} \mu(B \setminus A) \leq \mu(U_\delta \setminus A) < \delta.$$

- Since this holds for all $\delta > 0$, we have $\mu(A_\epsilon) = 0$.

□

Counterexamples

Definition

If (X, d) is a metric space, d is called *doubling* if for some $n \in \mathbb{N}$, any open ball $B_{2r}(x)$ can be covered by n balls of radius r.

Theorem (Käenmäki-Rajala-Suomala)

There is a finite Borel measure ν and a complete doubling metric δ on the Cantor space, compatible with the standard topology, such that some closed set C of positive measure has no ν-density points.

Theorem (Andretta-Costantini-Camerlo)

For any Polish measure space (X, d, μ) there is a compatible metric δ such that (X, δ, μ) does not satisfy Lebesgue's density theorem.

Question

Can the Lebesgue density theorem be generalized to other ideals instead of the ideal of null sets, in particular the σ-ideals defined by tree forcings?

Tree forcings and their ideals

$\star = 2$ or $\star = \omega$.

Definition

We say $\mathbb{P}$ is a *tree forcing* iff the conditions in $\mathbb{P}$ are perfect subtrees of ${}^{<\omega}\star$ ordered by inclusion such that for all $T \in \mathbb{P}$ and all $s \in T$ we have that $\{t \in T \mid s \subseteq t \text{ or } t \subseteq s\} \in \mathbb{P}$.

Definition (Ikegami)

Suppose that $\mathbb{P}$ is a tree forcing and A is a subset of ${}^{\omega}\star$.

(i) $A \in N_{\mathbb{P}}$ if for every $T \in \mathbb{P}$, there is some $S \in \mathbb{P}$ with $S \subseteq T$ and $[S] \cap A = \emptyset$. A set A in $N_{\mathbb{P}}$ is also called $\mathbb{P}$-null.

(ii) $I_{\mathbb{P}}$ is the σ-ideal generated by $N_{\mathbb{P}}$.

(iii) $A \in I^*_{\mathbb{P}}$ if for every $T \in \mathbb{P}$, there is some $S \in \mathbb{P}$ with $S \subseteq T$ and $[S] \cap A \in I_{\mathbb{P}}$.

Example: Random forcing

Definition

Let μ denote the uniform measure on ${}^{\omega}2$. Random forcing $\mathbb{B}$ is the tree forcing consisting of perfect trees $T \subseteq 2^{<\omega}$ such that $\mu([T]) > 0$ and for all $s \in T$, $\mu([\{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}]) > 0$.

Definition

A subset A of ${}^{\omega}2$ is $\mathbb{B}$-*measurable* iff for every $T \in \mathbb{B}$ there is some $S \in \mathbb{B}$ with $S \subseteq T$ such that either $[S] \cap A \in I_{\mathbb{B}}$ or $[S] \cap A^c \in I_{\mathbb{B}}$.

What is a $\mathbb{B}$-density point?

Density points for Random forcing

Definition

(i) Suppose that A is a $\mathbb{B}$-measurable subset of ${}^{\omega}2$. Suppose that $x \in {}^{\omega}2$. Then x is a $\mathbb{B}$-*translation density point* of A if for every $T \in \mathbb{B}$ and $s = \mathrm{stem}_T$, there is some n_0 such that for all $n \geq n_0$,

$$f_{x \upharpoonright n} \circ f_s^{-1}[T] \cap A \notin I_{\mathbb{B}}^*.$$

(ii) Let $D_{\mathrm{tr}}^{\mathbb{B}}(A)$ denote the set of translation density points of A.

(iii) We say that $\mathbb{B}$ has the *translation density property* if for every $\mathbb{B}$-measurable subset A of ${}^{\omega}2$, $A \triangle D_{\mathrm{tr}}^{\mathbb{B}}(A) \in I_{\mathbb{B}}^*$.

Here $f_s(x) = s^\frown x$ and $f_s^{-1}[T] = [T/s] = \{t \mid s^\frown t \in [T]\}$.

Density points for Random forcing

Lemma

Suppose that A is a $\mathbb{B}$-measurable subset of ${}^{\omega}2$.

(a) *If $\liminf_n \mu_n(x, A) = 1$, then x is a $\mathbb{B}$-translation density point of A.*

(b) *If $\liminf_n \mu_n(x, A) = 0$, then x is not a $\mathbb{B}$-translation density point of A.*

(c) *If $\liminf_n \mu_n(x, A) \in (0, 1)$, then x can be but does not have to be a $\mathbb{B}$-translation density point of A.*

Corollary

For every $\mathbb{B}$-measurable set A,

$$D_L(A) =_{I^*_{\mathbb{B}}} D^{\mathbb{B}}_{tr}(A).$$

In particular $\mathbb{B}$ has the translation density property.

Density points for ideals

Let I always denote a σ-ideal on the Borel subsets of ${}^{\omega}\star$.

Definition

(a) A map $g\colon A \to B$ between Borel sets A, B is I-*invariant* if for every Borel set $X \subseteq B$, we have $X \in I$ if and only if $g^{-1}[X] \in I$.

(b) Let $\mathrm{Bor}(I)$ denote the set of all I-invariant Borel isomorphisms $g\colon {}^{\omega}\star \to {}^{\omega}\star$.

(c) We say B is I-positive iff $B \notin I$. Let I^{+} denote the I-positive sets.

Density points for ideals

Definition

Suppose that A is a subset of ${}^{\omega}\star$ and Γ is a subgroup of $\mathrm{Bor}(I)$.

(i) An element x of ${}^{\omega}\star$ is an *I-density point of A w.r.t. Γ* if for every I-positive Borel set B, there is some $s \in \star^{<\omega}$ and some n_B such that for all $n \geq n_B$ and all $g \in \Gamma$,

$$(f_{x \restriction n} \circ g \circ f_s^{-1})[B] \cap A \notin I.$$

Let $D_{I,\Gamma}(A)$ denote the set of I-density points of A w.r.t. Γ.

(ii) An element x of ${}^{\omega}\star$ is a *strong I-density point* of A if there is some n_0 such that for all $n \geq n_0$, we have $f_{x \restriction n}^{-1}[A^c \cap N_{x \restriction n}] \in I$.
Let $D_{I,\mathrm{strong}}(A)$ denote the set of strong I-density points of A.

Note that $D_{I,\mathrm{strong}}(A) \subseteq D_{I,\mathrm{Bor}(I)}(A) \subseteq D_{I,\Gamma}(A) \subseteq D_{I,\{\mathrm{id}\}}(A)$.

Density points

Definition

We say that I has the *density property w.r.t.* Γ if for all Borel subsets A of ${}^{\omega}\star$,

$$A \triangle D_{I,\Gamma}(A) \in I.$$

Density points

Definition

(i) $\mathbb{P}$ is *homogeneous* if for all $S, T \in \mathbb{P}$, there is some $U \leq T$ and $f\colon [S] \to [U]$ in $\mathrm{Bor}(I^*_{\mathbb{P}})$.

(ii) $\mathbb{P}$ is *nondegenerate* if it is not equivalent to Cohen forcing and $\forall s \exists S \in \mathbb{P}\ s \subseteq \mathrm{stem}_S$.

(iii) (Friedman-Khomskii-Kulikov) A tree forcing $\mathbb{P}$ is *topological* if for all $S, T \in \mathbb{P}$ with $[S] \cap [T] \neq \emptyset$, there is $U \in \mathbb{P}$ with $[U] \subseteq [S] \cap [T]$. For a topological tree forcing $\mathbb{P}$, we let $\tau_{\mathbb{P}}$ be the topology on ${}^{\omega}\star$ with basis $\{[T] \mid T \in \mathbb{P}\}$.

Lemma

Assume that $\mathbb{P}$ is a homogeneous, nondegenerate, topological tree forcing and let $I = I^*_{\mathbb{P}}$. Then for all I-measurable A,

$$D_{I,\mathrm{strong}}(A) = D_{I,\mathrm{Bor}(I)}(A).$$

Density property

Definition

Suppose that $\mathbb{P}$ is a topological tree forcing and let A be a $I^*_{\mathbb{P}}$-measurable subset of ${}^{\omega}\star$ which has the property of Baire in $\tau_{\mathbb{P}}$. An $x \in {}^{\omega}\star$ is a $\mathbb{P}$-*topological density point* of A if $x \in U = D^{\mathbb{P}}_{\mathrm{top}}(A)$, where U is the unique open subset in $\tau_{\mathbb{P}}$ such that $A \triangle U \in I^*_{\mathbb{P}}$.

Theorem

Suppose that $\mathbb{P}$ is a homogeneous, nondegenerate, topological tree forcing with the ccc w.r.t. $I = I^*_{\mathbb{P}}$. Suppose that for every $T \in \mathbb{P}$ there is an $S \leq T$ such that every $x \in [S]$ is an I-density point of $[T]$ w.r.t. $\mathrm{Bor}(I)$. Then for every I-measurable A,

$$D_{I,\mathrm{Bor}(I)}(A) = D_{I,\mathrm{strong}}(A) =_I D^{\mathbb{P}}_{\mathrm{top}}(A).$$

Therefore I has the density property w.r.t. $\mathrm{Bor}(I)$.

Stem-linked forcings

Definition

Suppose that $\mathbb{P}$ is a tree forcing on ${}^{<\omega}\star$. Then $\mathbb{P}$ is *stem-linked* if for all $S, T \in \mathbb{P}$, if $\mathrm{stem}_S \subseteq \mathrm{stem}_T$ and $\mathrm{stem}_T \in S$, then S and T are compatible.

Stem-linked implies σ-linked, ccc w.r.t. $I^*_{\mathbb{P}}$, and topological.

Lemma

Let $\mathbb{P}$ be a stem-linked tree forcing on ${}^{<\omega}\star$. Then for every $T \in \mathbb{P}$ we have that every $x \in [T]$ is an $I^*_{\mathbb{P}}$-density point of $[T]$ w.r.t. $\mathrm{Bor}(I^*_{\mathbb{P}})$.

Corollary

Suppose $\mathbb{P}$ is a stem-linked, nondegenerate, homogeneous tree forcing on ${}^{<\omega}\star$. Then $I^*_{\mathbb{P}}$ has the density property w.r.t. $\mathrm{Bor}(I^*_{\mathbb{P}})$.

Examples

The I-density property w.r.t $\mathrm{Bor}(I)$ holds for the σ-ideals I defined by the following tree forcings, since they are stem-linked:

- Cohen forcing $\mathbb{C}$,
- Hechler forcing $\mathbb{H}$,
- F-Laver forcing $\mathbb{L}_F$ for a filter F, and
- F-Mathias forcing $\mathbb{R}_F$ for a filter F.

However, random forcing $\mathbb{B}$ does not have a dense stem-linked subset.

Counterexamples

The translation density property fails for the following tree forcings.

- Sacks forcing $\mathbb{S}$,
- Mathias forcing $\mathbb{R}$,
- Laver forcing $\mathbb{L}$,
- Miller forcing $\mathbb{M}$, and
- Silver forcing $\mathbb{V}$.

From ideals to forcings

Theorem (Ikegami)

Suppose that $\mathbb{P}$ *is a proper tree forcing. Then the map* $\iota\colon \mathbb{P} \to \mathcal{B}/I_{\mathbb{P}*}$ *that sends* $T \in \mathbb{P}$ *to the* $I_{\mathbb{P}*}$*-equivalence class represented by* $[T]$ *is a dense embedding, where* $\mathcal{B}$ *denotes the class of Borel subsets of* ${}^{\omega}\star$ *and* $\mathcal{B}/I_{\mathbb{P}*}$ *denotes the quotient Boolean algebra.*

Definition

A σ-ideal I on the Borel subsets of ${}^{\omega}\star$ is *homogeneous* if there are I-invariant Borel isomorphisms between any two I-positive Borel sets.

Open questions

Question

Does the density property fail for some homogeneous ccc σ-ideal?

Question

Does the density property hold for some homogeneous non-ccc σ-ideal?

Question

Is it consistent that there is no definable selector for the equivalence relation *equal modulo countable* on the class of Borel sets?