Mad families, definability, and ideals (Part 2)

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Overview

1. Definable mad families under Projective Determinacy

2. $\mathcal{J}$-mad families, for Borel ideals $\mathcal{J}$ other than Fin.

Definable mad families under Projective Determinacy

Theorem

Assuming the Axiom of Projective Determinacy (PD), there are no infinite projective mad families.

Our methods also give results under AD, but not all of them are new:

Theorem (Neeman-Norwood)

Assuming the Axiom of Determinacy holds in $L(\mathbb{R})$, there are no infinite mad families in $L(\mathbb{R})$.

We concentrate on PD. Our proofs use:

1. Every projective set is Suslin
2. Reasonably definable forcing cannot change the projective theory

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The analytic case

Theorem

There are no infinite Σ^1_1 mad families.

Proof ideas

- Suppose $\mathcal{A} = p[T]$ is an infinite a.d.-family.
- Let $\mathbb{M}_{\mathcal{A}^+}$ be Mathias forcing 'relative to $\mathcal{A}$'.
- $\Vdash_{\mathbb{M}_{\mathcal{A}^+}} \dot{x}_G$ is a.d. from every $x \in (p[T])^{V[G]}$.
- The formula Σ^1_2 formula

$$V[G] \models (\exists x)(\forall y \in \mathcal{A})\ x \neq y \wedge x \cap y \in Fin$$

is true in $V[G]$ and so by Shoenfied absoluteness, also in V.

This proof straightforwardly lifts to Σ^1_n using PD.

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Lemma

$\Vdash_{\mathbb{M}_{\mathcal{A}^+}} \dot{x}_G$ *is a.d. from every* $x \in (p[T])^{V[G]}$.

Note that it is easier to see that $\Vdash_{\mathbb{M}_{\mathcal{A}^+}} \dot{x}_G$ is a.d. from every $x \in (p[T])^V$, i.e., from $\mathcal{A}$.

Proof ideas

- Let $Z = \{x \in p[T] \mid x \cap x_G \notin Fin\}$.
- Show $|Z| \leq 1$.
- Supposing $Z \neq \emptyset$, let x be its unique element.
- x is definable from $[x_G]_{E_0}$.
- Show that $x \in V$, i.e., $x \in \mathcal{A}$. Contradiction!

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There are two non-trivial steps in the previous sketch:

1. $|\{x \in p[T] \mid x \cap x_G \notin Fin\}| \leq 1$.

   This uses heavily some property of the ideal Fin.

2. If a real x in $V[G]$ is definable from $[x_G]_{E_0}$, x is in V.

   This uses that
   - that $\mathcal{A}^+$ is σ-closed (uses that $\mathcal{A}$ is infinite!)
   - thus, $\mathbb{M}_{\mathcal{A}^+}$ is σ^*-closed in second component
   - that $\mathbb{M}_{\mathcal{A}^+}$ is homogeneous 'under finite changes'

Other ideals

Let $\mathcal{J}$ be an ideal on ω.

Two sets $A, A' \subseteq \omega$ are called *$\mathcal{J}$-almost disjoint iff $A \cap A' \in \mathcal{J}$*.

Let $\mathcal{J}^+ = \mathcal{P}(\omega) \setminus \mathcal{J}$

A set $\mathcal{A} \subset \mathcal{P}(\omega)$ is called a *$\mathcal{J}$-almost disjoint family iff $\mathcal{A} \subseteq \mathcal{J}^+$* and any two distinct sets in $\mathcal{A}$ are $\mathcal{J}$-almost disjoint.

$\mathcal{J}$-mad families are defined analogously.

Other Borel ideals

Martin Goldstern asked:

Question:

Is there an analytic $\mathcal{J}$-mad family where $\mathcal{J}$ is the harmonic ideal:

$$X \in \mathcal{J} \iff \sum_{n \in X} 1/n < \infty$$

The answer in this case is no; and as before the proof lifts under PD (and lifts further under AD).

Fix a σ^*-closed Borel ideal $\mathcal{J}$ and a Suslin infinite $\mathcal{J}$-a.d. family $\mathcal{A} \subseteq \mathcal{P}(\omega)$, $\mathcal{A} = p[T]$. Denote $(\mathcal{J}, \mathcal{A})^+$ by the co-ideal of the ideal generated by $\mathcal{A} \cup \mathcal{J}$.

Lemma

$(\mathcal{J}, \mathcal{A})^+$ is σ^*-closed.

The obvious forcing $\mathbb{M}_{(\mathcal{J},\mathcal{A})^+}$ is

- homogeneous under changes in $\mathcal{J}$
- and σ^*-closed in the second part

We need more to show two more crucial properties:

- $\Vdash_{\mathbb{M}_{(\mathcal{J},\mathcal{A})^+}} \dot{x}_G \notin \mathcal{J}^{V[G]}$
- $\Vdash_{\mathbb{M}_{(\mathcal{J},\mathcal{A})^+}} \{x \in p[T] \mid \dot{x}_G \cap x \in \mathcal{J}^{V[G]}\}$ has at most one element.

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Submeasures and ideals

A *submeasure* is a function

$$\Phi\colon \mathcal{P}(\omega) \to [0, \infty]$$

such that

- $X \subseteq Y \Rightarrow \Phi(X) \leq \Phi(Y)$ for any $X, Y \in \mathcal{P}(\omega)$
- $\Phi(X \cup Y) \leq \Phi(X) + \Phi(Y)$ for any $X, Y \in \mathcal{P}(\omega)$
- $\Phi(\{n\}) < \infty$ for any $n \in \omega$
- Φ is *lower semi-continuous*. I.e.,

$$\Phi(X) = \lim_n \Phi(X \cap n)$$

Any submeasure gives rise to an ideal $Fin(\Phi)$ by letting

$$Fin(\Phi) = \{X \in \mathcal{P}(\omega) \mid \Phi(X) < \infty\}$$

Generalization to finite part of submeasure

The previous arguments go through for ideals of the form $Fin(\Phi)$ where Φ is a submeasure. In particular this answers Goldstern's question.

If $\mathcal{J} = Fin(\Phi)^+$

- $\mathcal{J}$ is σ^*-closed.
- For any $\mathcal{J}$-a.d. family $\mathcal{A}$, $\mathbb{M}_{\mathcal{J},\mathcal{A}^+}$ is 'homogeneous under changes on sets in $\mathcal{J}$' and σ^*-closed on the second part.
- $\Vdash_{\mathbb{M}_{\mathcal{J},\mathcal{A}^+}} \phi(\dot{x}_G) = \infty$, i.e., $\dot{x}_G \notin \mathcal{J}^{V[G]}$
- $\Vdash_{\mathbb{M}_{\mathcal{J},\mathcal{A}^+}} \{x \in p[T] \mid \dot{x}_G \cap x \in \mathcal{J}^{V[G]}\}$ has at most one element.

What about ideals which are not of this form?

Let J be any ideal. Define an ideal J^ω on $\omega \times \omega$ by:

$$X \in J^\omega \iff (\forall n \in \omega)\ X(n) \in J$$

That is, J^ω the ideal generated by sets of the form

$$\bigcup_{n\in\omega} \{n\} \times J_n,$$

where $(J_n)_{n\in\omega}$ is any sequence from $\mathcal{J}$.

Observation

Let $\mathcal{J}$ be any non-trivial ideal. There is a countable $\mathcal{J}^\omega$-mad family, namely $\{\{n\} \times \omega \mid n \in \omega\}$.

In particular such ideals appear cofinally in the Borel hierarchy.

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Fubini product ideals

Consider the ideal $\mathrm{Fin} \times \mathrm{Fin}$ on $\omega \times \omega$, defined as follows:
For $X \subseteq \omega \times \omega$ (letting $X(n) = \{m \mid (n, m) \in X\}$), define

$$X \in \mathrm{Fin} \times \mathrm{Fin} \iff \{n \in \omega \mid X(n) \notin \mathrm{Fin}\} \in \mathrm{Fin}$$

- Using 'Fubini products', we can define Fin^α, for every $\alpha < \omega_1$.
- Each Fin^α is a definable ideal (in fact, Borel).
- The Fin^α appear cofinally in the Borel hierarchy.

Theorem

For each $\alpha < \omega_1$,

- *There is no analytic Fin^α-mad family*
- *Under the Axiom of Projective Determinacy, there is no infinite projective Fin^α-mad family*
- *Under the Axiom of Determinacy, there is no infinite Fin^α-mad family in $L(\mathbb{R})$.*

Question:

Can we characterize the analytic, or at least the Borel ideals $\mathcal{J} \subseteq \mathcal{P}(\omega)$ such that there is no analytic infinite $\mathcal{J}$-mad family?

Grazie mille!