Hyperfiniteness of boundary actions of hyperbolic groups

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Tail equivalence is the equivalence relation on 2^ω generated by $x \sim \mathrm{shift}(x)$. In other words,

$$x \sim y \iff \exists k, l \forall n [x_{k+n} = y_{l+n}]$$

There is a natural example of tail equivalence arising in the context of free groups.

Denote the free group by $F_2 = \langle a, b \rangle$. Recall the Cayley graph. An infinite path from the origin is called a **geodesic ray**. Note that every geodesic ray can be represented by an infinite sequence in $\{a, a^{-1}, b, b^{-1}\}$.

The set of geodesic rays is called the **boundary** of F_2 and is denoted by ∂F_2. Note that F_2 acts on ∂F_2 by concatenation (prepending the group element to the geodesic ray). The orbit equivalence relation, denoted $E_{F_2}^{\partial F_2}$, is just tail equivalence.

By a classical result of Dougherty-Jackson-Kechris, tail equivalence is hyperfinite. Thus by the above discussion, we have:

Theorem

$E_{F_2}^{\partial F_2}$ is a hyperfinite Borel equivalence relation.

We'd like to generalize this theorem to more general groups G. To do this, we need a boundary ∂G with a G-action and a Polish topology. Hyperbolic groups fit the bill.

Conjecture

Let G be a hyperbolic group. Then $E_G^{\partial G}$ is hyperfinite.

Intuitive definitions:

Definition

Let X be a metric space.

- A **geodesic** is an isometric embedding of $[a, b]$ into X.
- A **geodesic triangle** is a triangle whose sides are geodesics.

To define hyperbolicity, we will use the idea of slim triangles:

Definition (slim triangles)

A geodesic triangle is δ-**slim** if every side is contained in the closed δ-nhd of the union of the other two sides.

Definition (Rips)

Let X be a geodesic metric space.

- X is δ-**hyperbolic** ($\delta \geq 0$) if every geodesic triangle in X is δ-thin.
- X is **hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$.

Example

- Trees (0-hyperbolic)
- Hyperbolic space
- Closed hyperbolic manifolds

Definition

Let G be a group with a finite generating set S. Then G is **hyperbolic** if $\mathrm{Cay}(G, S)$ is hyperbolic.

Remark

The above definition is technically a definition of (G, S) being hyperbolic, but it is in fact independent of the generating set.

Example

- Free group
- π_1 of closed hyperbolic manifolds

Now we need the notion of a boundary.

Definition

A **geodesic ray** is an isometric embedding of $[0, \infty)$.

We can have two geodesic rays which converge to the same point on the boundary.

Definition

For a hyperbolic space X, the **Gromov boundary** of X, denoted ∂X, is the quotient by Hausdorff distance of the set of all geodesic rays in X.

Example

- The boundary of an interesting tree is a Cantor space.
- $\partial \mathbb{H}^n = S^n$.

There is a Polish topology on ∂X (coming from a uniform structure on the geodesic rays).
Now if a group G acts on X, then it induces an action on ∂X.
Thus our conjecture from before makes sense:

Conjecture

Let G be a hyperbolic group.
Then $E_G^{\partial G}$ is hyperfinite.

The thing to try is to emulate the original proof of Dougherty-Jackson-Kechris.

Proposition

Let G be a hyperbolic group and fix a Cayley graph $\mathrm{Cay}(G, S)$.
Suppose that $[x, a) \triangle [y, a)$ is finite for all $x, y \in \mathrm{Cay}(G, S)$ and $a \in \partial G$.
Then $E_G^{\partial G}$ is hyperfinite.

Here, $[x, a)$ denotes the following set:

$$[x, a) := \{z \in \mathrm{Cay}(G, S) : z \text{ lies on a geodesic from } x \text{ to } a\}$$

$[x, a) \triangle [y, a)$ is finite for all $x, y \in \mathrm{Cay}(G, S)$ and $a \in \partial G$.

Question

Does every Cayley graph satisfy this condition?

Answer

No, even free groups can have bad Cayley graphs (Nicholas Touikan, 2017).

It's open whether every group has a good Cayley graph or not. However, we can relax our conditions:

Proposition

Let G be a hyperbolic group acting geometrically on a locally finite graph X.
Suppose that $[x, a) \triangle [y, a)$ is finite for all $x, y \in X$ and $a \in \partial X$. Then $E_G^{\partial G}$ is hyperfinite.

$[x, a) \triangle [y, a)$ is finite for all $x, y \in X$ and $a \in \partial X$.

Question

Which graphs satisfy this condition?

Theorem (Huang-Sabok-S)

Locally finite hyperbolic CAT(0) *cube complexes satisfy the condition.*

Corollary (Huang-Sabok-S)

Let G be a hyperbolic group acting geometrically on a CAT(0) *cube complex.*
Then $E_G^{\partial G}$ is hyperfinite.

What's a CAT(0) cube complex?

Definition

A **cube complex** is a polygonal complex built out of Euclidean cubes.

Definition

The **link** of a vertex v on a cube complex X is the simplicial complex obtained by intersecting X with an ϵ-sphere centered at v.

Definition

A simplicial complex is **flag** if every clique spans a simplex.

Definition

A CAT(0) **cube complex** is a cube complex which is simply connected and whose vertex links are flag.

Remark

The flag vertex links guarantee nonpositive curvature.

Theorem (Huang-Sabok-S)

Let X be a locally finite hyperbolic CAT(0) *cube complex.*
Then $[x, a) \triangle [y, a)$ is finite for any $x, y \in X$ and $a \in \partial X$.

Corollary (Huang-Sabok-S)

Let G be a hyperbolic group acting geometrically on a CAT(0) *cube complex (known as a* **cubulated** *group).*
Then $E_G^{\partial G}$ is hyperfinite.

Examples of cubulated hyperbolic groups:

- Free groups
- Surface groups (this can be seen by dividing up the polygon)
- Hyperbolic closed 3-manifold groups (Kahn-Markovic and Bergeron-Wise)
- Gromov random groups (of density $< \frac{1}{6}$)

(*) $[x, a) \triangle [y, a)$ is finite for all $x, y \in X$ and $a \in \partial X$.
The following is still open:

Conjecture

Every hyperbolic group G acts geometrically on a locally finite graph X satisfying (*).

Theorem (Timothée Marquis, 2017)

Locally finite hyperbolic buildings satisfy (*).

Thank you!