Convergence of a typical martingale

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Martingales – definition

$$\mathcal{M} = \left\{ f \in \ell_\infty(2^{<\omega});\ \forall s \in 2^{<\omega} \colon f(s) = \tfrac{1}{2}\left(f(s^\wedge 0) + f(s^\wedge 1)\right) \right\}.$$

Let μ denote the usual product measure on 2^ω.

Theorem (Doob Convergence Theorem)

Every bounded martingale $f \in \mathcal{M}$ converges for μ-a.a. $\sigma \in 2^\omega$, i.e., $\lim_{n\to\infty} f(\sigma|_n)$ exists for μ-a.a. $\sigma \in 2^\omega$.

- Measure point of view
- Topological point of view

$$\mathcal{M}_1 = \{f \in \mathcal{M};\ \forall s \in 2^{<\omega}\colon 0 \leq f(s) \leq 1\}.$$

The measure Λ on the space of martingales

- $\pi_n\colon \bigcup_{n<j\leq\omega} \mathbb{R}^{2^{<j}} \to \mathbb{R}^{2^{\leq n}}$
- $\mathcal{M}_1(n) := \pi_n(\mathcal{M}_1)$
- $\psi_n\colon \mathbb{R}^{2^{\leq n}} \to \mathbb{R}^{2^n}$; $\psi_n(x) = (x_s)_{s\in 2^n}$
- For $f \in \mathbb{R}^{2^{\leq n}}$ and $x \in \mathbb{R}^{2^{n+1}}$ we set

$$f^\wedge x(s) = \begin{cases} f(s), & s \in 2^{\leq n}, \\ x(s), & s \in 2^{n+1}. \end{cases}$$

- For $\mathcal{A} \subset \mathcal{M}_1(n+1)$ and $f \in \mathcal{M}_1(n)$ we set

$$\mathcal{A}^f = \{x \in \mathbb{R}^{2^{n+1}};\ f^\wedge x \in \mathcal{A}\}.$$

- $A(\alpha) = \{(u,v) \in [0,1]^2;\ \frac{1}{2}(u+v) = \alpha\}$, $\alpha \in \mathbb{R}$
- For $\alpha \in [0,1]$ we set

$$\zeta_\alpha = \begin{cases} \varepsilon_{[0,0]} & \text{for } \alpha = 0, \\ \frac{1}{\mathcal{H}_1(A(\alpha))}\mathcal{H}_1|_{A(\alpha)} & \text{for } \alpha \in (0,1), \\ \varepsilon_{[1,1]} & \text{for } \alpha = 1. \end{cases}$$

- The mapping $\alpha \mapsto \zeta_\alpha$ is continuous from $[0,1]$ to the space of Radon measures on $[0,1]^2$ equipped with the weak topology.
- For $x \in [0,1]^{2^n}$ we set $\Delta_x := \prod_{s \in 2^n} \zeta_{x(s)}$ on $[0,1]^{2^{n+1}}$.

Definition of $(\Lambda_n)_{n\in\omega}$

- Λ_0 = the restriction of the Lebesgue measure on $\mathcal{M}_1(0)(= [0,1])$
- $$\Lambda_{n+1}(\mathcal{A}) = \int_{\mathcal{M}_1(n)} \Delta_{\psi_n(f)}(\mathcal{A}^f)\,\mathrm{d}\Lambda_n(f), \quad \mathcal{A} \subset \mathcal{M}_1(n+1) \text{ Borel.}$$

Since the mapping $f \mapsto \Delta_{\psi_n(f)}$ is continuous from $\mathcal{M}_1(n)$ to the set of probability measures on $[0,1]^{2^{n+1}}$, the measure Λ_{n+1} is a well defined Radon measure. Observe that $\pi_n(\Lambda_{n+1}) = \Lambda_n$.

Λ on $\mathcal{M}_1 =$ the inverse limit of $(\Lambda_n)_{n\in\omega}$(Prokhorov Theorem)

Theorem

Denote

$$\mathcal{M}^* = \{f \in \mathcal{M}_1;\ f(\sigma) \in \{0,1\} \text{ for } \mu\text{-a.a. } \sigma \in 2^\omega\}.$$

Then $\Lambda(\mathcal{M}^*) = 1$.

Lemma (Chernoff)

Let $m, k \in \omega$*,* $m > 1$*,* $p \in (0, 1]$*, and* $k \leq mp$*. Then we have*

$$\sum_{j=0}^{k} \binom{m}{j} p^j (1-p)^{m-j} \leq \exp\Big(-\frac{(mp-k)^2}{2pm}\Big).$$

A possible strengthening of Theorem 2 saying that for Λ-a.a. martingales f we have $f(\sigma) \in \{0, 1\}$ for every $\sigma \in 2^\omega$ does not holds.

Proposition

For Λ-a.a. martingales $f \in \mathcal{M}_1$ there exists $\sigma \in 2^\omega$ such that $f(\sigma)$ is not defined or $f(\sigma) \in (0, 1)$.

Topological point of view

Theorem

The set

$$\mathcal{M}^\sharp = \{f \in \mathcal{M}_1;\ D_f \textit{ is comeager in } 2^\omega\}$$

is comeager in $\mathcal{M}_1$.

For $f \in \mathcal{M}$ and $\sigma \in 2^\omega$ we set

$$\operatorname{osc}(f|_\sigma) := \inf_{n\in\omega} \sup_{j,k\geq n} |f(\sigma|_j) - f(\sigma|_k)|.$$

Lemma

Let $f \in \mathcal{M}_1$. Then

(a) *the set $\{\sigma \in 2^\omega;\ \operatorname{osc}(f|_\sigma) \geq \alpha\}$ is G_δ for every $\alpha \in \mathbb{R}$,*

(b) *the set D_f is Borel in 2^ω.*

Lemma

For any $r > 0$ *there exists* $f \in \mathcal{M}$ *such that*

- $\forall s \in 2^{<\omega}\colon\ -r < f(s) < r$,
- *the set* $\{\sigma \in 2^{\omega};\ \operatorname{osc}(f|_{\sigma}) = 2r\}$ *is comeager in* 2^{ω}.

Let τ_p be the topology of pointwise convergence.

Theorem

The set

$$\mathcal{M}' = \{f \in \mathcal{M}_1;\ \{\sigma \in 2^\omega;\ \operatorname{osc}(f|_\sigma) = 1\} \text{ is comeager}\}$$

is comeager in $(\mathcal{M}_1, \tau_p)$.