Dropping Polishness

Andrea Medini

Kurt Gödel Research Center
University of Vienna

March 6, 2015

How do we "generalize" descriptive set theory?

Recall that a space is *Polish* if it is separable and completely metrizable. I can think of three ways...

1. Dropping separability: the prototypical space is κ^ω, where κ has the discrete topology. It is completely metrizable, but not necessarily separable. (Ask Sergey Medvedev...)
2. Dropping everything: the prototypical space is κ^κ with the $<\kappa$-box-topology, where κ has the discrete topology and it satisfies $\kappa^{<\kappa} = \kappa$. (Ask Sy Friedman...)
3. Dropping Polishness: consider questions of "descriptive set-theoretic flavor" in spaces that are separable and metrizable, but not necessarily **completely** metrizable. (Ask Arnie Miller...)

From now on, we will assume that every space is separable and metrizable, but not necessarily Polish.

How do you define complexity then?

Γ will always be one of the following (boldface) pointclasses.

- $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$, where ξ is an ordinal such that $1 \leq \xi < \omega_1$ (these are the *Borel pointclasses*).
- $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$, where n is an ordinal such that $1 \leq n < \omega$ (these are the *projective pointclasses*).

We will assume that the definition of a Γ subset of a Polish space is well-known, and recall that it can be generalized to arbitrary spaces as follows.

Definition

Fix a pointclass Γ. Let X be a space. We will say that $A \subseteq X$ is a Γ *subset of* X if there exists a Polish space T containing X as a subspace such that $A = B \cap X$ for some Γ subset B of T.

In the case of the Borel pointclasses, this is not really necessary, because the usual definition works in arbitrary spaces. But we prefer to give a unified treatment.

The following “reassuring” proposition can be proved by induction on Γ.

Proposition

Fix a pointclass Γ. Let X be a space and $A \subseteq X$. Then the following conditions are equivalent.

- *A is a Γ subset of X.*
- *For every space T containing X as a subspace there exists a Γ subset B of T such that $A = B \cap X$.*

One could also define the so-called *ambiguous pointclasses* as follows.

- Let $\Delta^0_\xi = \Sigma^0_\xi \cap \Pi^0_\xi$ for every ordinal ξ such that $1 \leq \xi < \omega_1$.
- Let $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$ for every ordinal n such that $1 \leq n < \omega$.

However, they would not add any interesting content to our results. Therefore, we will exclude them from our discussion. The only exception is $\Delta^0_1 =$ clopen.

Illustrious precedents

Definition

The *Baire order* $\mathrm{ord}(X)$ of a space X is the minimum ordinal α such that $\Sigma^0_\alpha(X) = \mathrm{Borel}(X)$.

Examples:

- $\mathrm{ord}(X) = 1$ if and only if X is discrete.
- $\mathrm{ord}(X) \leq 2$ whenever X is countable, and $\mathrm{ord}(\mathbb{Q}) = 2$.
- $\mathrm{ord}(X) \leq \omega_1$ for every space.
- $\mathrm{ord}(X) = 3$ if X is a Luzin set. (Recall that an uncountable set of reals X is a *Luzin set* if every uncountable subset of X is non-meager. They exist under CH.)
- $\mathrm{ord}(X) = 2$ if X is a Sierpiński set. (Recall that an uncountable set of reals X is a *Sierpiński set* if every uncountable subset of X is non-null. They exist under CH.)

Theorem (Lebesgue)

$\mathrm{ord}(X) = \omega_1$ *for every uncountable Polish space* X.

Banach asked whether we can "drop Polishness"...

Conjecture (Banach)

$\mathrm{ord}(X) = \omega_1$ *for every uncountable space* X.

Theorem (Miller, 1979)

It is consistent that $\mathrm{ord}(X) = \omega_1$ *for every uncountable space* X.

Theorem (Kunen)

Assume CH*. Then for every* α *such that* $1 \leq \alpha < \omega_1$ *there exists a space* X *such that* $\mathrm{ord}(X) = \alpha$.

Miller improved this by showing that CH can be weakened to the existence of a Luzin set.

Perfect set property: the classical case

Definition

Let X be a space and Γ a pointclass. We will say that X has the *perfect set property for* Γ *subsets* (briefly, the $\mathrm{PSP}(\Gamma)$) if every Γ subset of X is either countable or it contains a copy of 2^ω.

One of the classical problems of descriptive set theory consists in determining for which pointclasses Γ the statement "Every Polish space has the $\mathrm{PSP}(\Gamma)$" holds. The following three famous theorems essentially solve this problem.

Theorem

- *(Suslin) Every Polish space has the* $\mathrm{PSP}(\mathrm{analytic})$.
- *(Gödel) Assume* $\mathrm{V} = \mathrm{L}$. *Then no uncountable Polish space has the* $\mathrm{PSP}(\mathrm{coanalytic})$.
- *(Davis) Assume the axiom of Projective Determinacy. Then every Polish space has the* $\mathrm{PSP}(\mathrm{projective})$.

Perfect set property: the non-Polish case

But what happens in arbitrary (that is, not necessarily Polish) spaces? By the following simple proposition, the problem described above becomes trivial.

Recall that a subset B of an uncountable Polish space T is a *Bernstein set* if $B \cap K \neq \varnothing$ and $(T \setminus B) \cap K \neq \varnothing$ for every copy K of 2^ω in T. It is easy to see that Bernstein sets exist in ZFC. Since $2^\omega \approx 2^\omega \times 2^\omega$, every Bernstein set has size $\mathfrak{c}$.

Proposition

Let X be a Bernstein set in some uncountable Polish space. Then X does not have the $\mathsf{PSP}(\Gamma)$ *for any poinclass* Γ.

Proof: Let Γ be a pointclass. Then X itself is an uncountable Γ subset of X that does not contain any copy of 2^ω.

Thank you for your attention

and good night!

Perfect set property: the non-Polish case, II

Much less trivial, however, is to determine the status of the statement

> *"For every space X, if X has the* PSP(Γ) *then X has the* PSP(Γ')*"*

as Γ, Γ' range over all pointclasses of complexity at most analytic. We will give a complete solution to this problem. The case "PSP(analytic) vs. PSP(closed)" turned out to be the most interesting one... Do you think they're equivalent or not?

Theorem

The following are equivalent.

- *For every space X, if X has the* PSP(closed) *then X has the* PSP(analytic)*.*
- $\mathfrak{b} > \omega_1$.

We will prove the interesting half of the above theorem.

A first attempt (suggested by Kunen)

Assume $\mathfrak{b} = \omega_1$. We want to construct a space X such that

- Every uncountable closed subset of X contains a copy of 2^ω, but
- There exists an uncountable analytic subset of X that contains no copies of 2^ω.

Using $\mathfrak{b} = \omega_1$, fix a family $Z = \{f_\alpha : \alpha \in \omega_1\} \subseteq \omega^\omega$ such that

- Z is unbounded (there exists no $g \in \omega^\omega$ such that $f <^* g$ for all $f \in Z$),
- Z is well-ordered ($f_\alpha <^* f_\beta$ whenever $\alpha < \beta$).

X will be a subspace of $T = (\omega + 1)^\omega \approx 2^\omega$, where $\omega + 1 = \omega \cup \{\omega\}$ is the converging sequence with limit ω. Also define $T_n = \{x \in T : x(n) = \omega\}$ for every $n \in \omega$, so that

$$T = \omega^\omega \cup \bigcup_{n \in \omega} T_n.$$

Define

$$X = Z \cup \bigcup_{n\in\omega} T_n \subseteq (\omega+1)^\omega$$

Lemma

Let Z be an unbounded family. Then

- $\mathrm{cl}(Z)$ *is not compact, where the closure is taken in* ω^ω,
- $\mathrm{cl}(Z) \not\subseteq \omega^\omega$, *where the closure is taken in* $(\omega+1)^\omega$.

Lemma

Let Z be a **well-ordered** *unbounded family of size ω_1.*
Then, for every uncountable $Y \subseteq Z$,

- $\mathrm{cl}(Y)$ *is not compact, where the closure is taken in* ω^ω,
- $\mathrm{cl}(Y) \not\subseteq \omega^\omega$, *where the closure is taken in* $(\omega+1)^\omega$.

Since each $T_n \approx (\omega+1)^\omega$ is compact, Z is a G_δ subset of X. So, by the second lemma, Z witnesses that X does not have the $\mathrm{PSP}(\mathrm{G}_\delta)$. In particular, X does not have the PSP(analytic).

What goes wrong? (And how do we fix it?)

We still have to show that every uncountable closed subset of X contains a copy of 2^ω.

Dream

Is it true that $\mathrm{cl}(Y) \cap \bigcup_{n\in\omega} T_n$ *is uncountable whenever* $Y \subseteq Z$ *is uncountable?*

Reality

If Z *consists only of increasing functions, then*

$$\mathrm{cl}(Z) \cap \bigcup_{n\in\omega} T_n \subseteq \{s^\frown\langle\omega,\omega,\ldots\rangle : s \in \omega^{<\omega}\},$$

which is countable!

The solution comes from a property of certain subsets of 2^ω. Since the speaker is just a romantic little guy, he decided to name it after the lovely area of Vienna where he lives...

Introducing

the Grinzing property!

The Grinzing property

Definition

We will say that a subset W of 2^ω has the *Grinzing property* (briefly, the GP) if it is uncountable and for every uncountable $Y \subseteq W$ there exist uncountable subsets Y_α of Y for $\alpha \in \omega_1$ such that $\mathrm{cl}(Y_\alpha) \cap \mathrm{cl}(Y_\beta) = \varnothing$ whenever $\alpha \neq \beta$, where the closure is taken in 2^ω.

Notice that an uncountable $W \subseteq 2^\omega$ has the GP if and only if every subset of W of size ω_1 has the GP.
Could the whole 2^ω have the GP?

Theorem

- *Assume* CH. *Then* 2^ω *does not have the* GP.
- *Assume* $\mathrm{MA} + \neg\mathrm{CH}$. *Then* 2^ω *has the* GP.

Why do we care?

As we have just seen, it is consistent that **all** uncountable subsets of 2^ω have the GP. Therefore, it seems likely that there exists at least **one** in ZFC...

Question

Is it possible to prove in ZFC *that there exists a subset of* 2^ω *with the* GP*?*

This would allow us to finish the proof of the main theorem!
Let W be a subset of 2^ω with the GP such that $|W| = \omega_1$.
Fix an injective enumeration $W = \{g_\alpha : \alpha \in \omega_1\}$.

	Old "proof"	**New proof**
T	$(\omega+1)^\omega$	$(\omega+1)^\omega \times 2^\omega$
T_n	$\{x \in (\omega+1)^\omega : x(n) = \omega\}$	$\{x \in (\omega+1)^\omega : x(n) = \omega\} \times 2^\omega$
Z	$\{f_\alpha : \alpha \in \omega_1\}$	$\{\langle f_\alpha, g_\alpha \rangle : \alpha \in \omega_1\}$

Notice that each $T_n \approx (\omega+1)^\omega \times 2^\omega \approx 2^\omega$ as before.

Define

$$X = Z \cup \bigcup_{n \in \omega} T_n \subseteq (\omega + 1)^\omega \times 2^\omega$$

The PSP(analytic) still fails
As before, Z is a G_δ subset of X. If Z contained a copy of 2^ω, then $\pi[Z]$ would too, where $\pi : \omega^\omega \times 2^\omega \longrightarrow \omega^\omega$ is the projection on the first coordinate.

Thanks to the GP, we can finally prove the PSP(closed)
As before, we have to prove that $\mathrm{cl}(Y) \cap \bigcup_{n \in \omega} T_n$ is uncountable for every uncountable $Y \subseteq Z$. The difference is that now, it **will** be enough to prove $\mathrm{cl}(Y) \cap \bigcup_{n \in \omega} T_n \neq \varnothing$ for every such Y.
In fact, Y will be in the form $\{\langle f_\alpha, g_\alpha \rangle : \alpha \in S\}$ for some uncountable $S \subseteq \omega_1$. Then look at

$$\{g_\alpha : \alpha \in S\} \subseteq W$$

and use the fact that W has the GP.

OK then. Why is it non-empty?

Fix an uncountable $Y \subseteq Z$. We have to prove that $\mathrm{cl}(Y) \cap \bigcup_{n\in\omega} T_n \neq \varnothing$.

Since being a well-ordered unbounded family is preserved by taking uncountable subsets, we can assume that $Y = Z$.

By an old lemma, there exists $f \in (\omega+1)^\omega \setminus \omega^\omega$ and a sequence $\langle \alpha_n : n \in \omega \rangle$ of elements of ω_1 such that

$$\langle f_{\alpha_n} : n \in \omega \rangle \longrightarrow f$$

Since 2^ω is compact, there exists $g \in 2^\omega$ and a subsequence of $\langle g_{\alpha_n} : n \in \omega \rangle$ that converges to g in 2^ω.

It follows that the corresponding subsequence of $\langle \langle f_{\alpha_n}, g_{\alpha_n} \rangle : n \in \omega \rangle$ converges to $\langle f, g \rangle$, which is clearly an element of $\bigcup_{n\in\omega} T_n$.

A set with the Grinzing property in ZFC

Theorem (Todorčević)

There exists a subtree $\mathcal{T}$ of $\omega^{<\omega_1}$ and a system $\langle K_s : s \in \mathcal{T}\rangle$ of perfect subsets of 2^ω satisfying the following properties.

- *Each level of $\mathcal{T}$ is countable and non-empty.*
- $K_t \subsetneq K_s$ *whenever* $t \supsetneq s$.
- $K_s \cap K_t = \varnothing$ *whenever* $s \perp t$.

Since there are no strictly descending ω_1-sequences of closed subsets of 2^ω, any tree $\mathcal{T}$ as above must be Aronszajn.
Miller found a mistake in my "proof" of the following result, and kindly supplied a new one, based on the above theorem.

Corollary (Miller)

There exists a subset of 2^ω with the GP.

Proof of Miller's result

Let $\mathcal{T}$ and $\langle K_s : s \in \mathcal{T} \rangle$ be given by Todorčević's theorem. Let $W = \{w_s : s \in \mathcal{T}\}$ where each $w_s \in K_s$ and they are distinct. We will show that W has the GP.

Fix an uncountable $Y \subseteq W$ and let $\mathcal{S}$ be the subtree of $\mathcal{T}$ generated by $\{s \in \mathcal{T} : w_s \in Y\}$. Assume without loss of generality that $\{t \in \mathcal{S} : t \supseteq s\}$ is uncountable for every $s \in \mathcal{S}$.

Notice that $\mathcal{S}$ cannot be Souslin, otherwise forcing with $\mathcal{S}$ would yield a strictly descending ω_1-sequence of closed subsets of 2^ω, contradicting the fact that ω_1 is preserved.

So we can fix an uncountable antichain $\langle s_\alpha : \alpha \in \omega_1 \rangle$ in $\mathcal{S}$. It is easy to check that setting $Y_\alpha = Y \cap K_{s_\alpha}$ for $\alpha \in \omega_1$ yields uncountable subsets of Y with pairwise disjoint closures in 2^ω.

No finer distinctions are possible

At this point, it seems natural to ask whether finer distinctions are possible. Consider for example the following questions.

- Is it consistent that there exists a space with the PSP(Borel) but not the PSP(analytic)?
- Is it consistent that there exists a space with the PSP(G_δ) but not the PSP(Π^0_3)?

The following result shows that the answer to such questions (and several other variants of them) is “no”.

Theorem (Solecki, 1994)

Let $\mathcal{I}$ be a family of closed subsets of a Polish space T. Let A be an analytic subset of T. Then one of the following holds.

1. *$A \subseteq \bigcup \mathcal{J}$ for some countable $\mathcal{J} \subseteq \mathcal{I}$.*
2. *There exists a G_δ subset G of T such that $G \subseteq A$ and $G \nsubseteq \bigcup \mathcal{J}$ for every countable $\mathcal{J} \subseteq \mathcal{I}$.*

Corollary

Let X be a space with the PSP(G_δ)*. Then X has the* PSP(analytic)*.*

Proof: Without loss of generality, assume that X is a subspace of $T = [0,1]^\omega$. Define

$$\mathcal{I} = \{K \subseteq T : K \text{ is closed in } T \text{ and } |X \cap K| < \omega_1\}.$$

Let A be an analytic subset of T such that $A \cap X$ is uncountable. Since ω_1 has uncountable cofinality, condition (1) cannot hold. Therefore, condition (2) must hold for some G_δ subset G of T. In particular $G \cap X$ is uncountable, so it contains a copy of 2^ω by the PSP(G_δ). Since $G \subseteq A$, it follows that $A \cap X$ contains a copy of 2^ω, which is what we needed to show.

The complete picture

Theorem

Consider the following conditions on a space X.

1. *X has the* PSP(analytic).
2. *X has the* PSP(G_δ).
3. *X has the* PSP(F_σ).
4. *X has the* PSP(closed).
5. *X has the* PSP(open).
6. *X has the* PSP(clopen).

The implications $(1) \leftrightarrow (2) \rightarrow (3) \leftrightarrow (4) \rightarrow (5) \rightarrow (6)$ *hold for every space X. The implication* $(2) \leftarrow (3)$ *holds for every space X if and only if $\mathfrak{b} > \omega_1$. There exist* ZFC *counterexamples to the implications* $(4) \leftarrow (5)$ *and* $(5) \leftarrow (6)$.

Generalizing the Grinzing property

Definition

Fix cardinals κ, λ such that $\omega_1 \leq \kappa \leq \mathfrak{c}$ and $\lambda \leq \kappa$. We will say that a subset W of 2^ω has the (κ, λ)-*Grinzing property* (briefly, the (κ, λ)-GP) if $|W| \geq \kappa$ and for every $Y \subseteq W$ such that $|Y| \geq \kappa$ there exist subsets Y_α of Y for $\alpha \in \lambda$ such that $|Y_\alpha| \geq \kappa$ for each α and $\mathrm{cl}(Y_\alpha) \cap \mathrm{cl}(Y_\beta) = \varnothing$ whenever $\alpha \neq \beta$, where the closure is taken in 2^ω.

Just like in the case of the ordinary GP, a subset W of 2^ω of size at least κ has the (κ, λ)-GP if and only if every subset of W of size κ has the (κ, λ)-GP.
Also, it is clear that the (κ, λ)-GP gets stronger as λ gets bigger.
Furthermore, the (ω_1, ω_1)-GP is simply the GP.

It is easy to show that 2^ω has the (κ, ω)-GP for every cardinal $\kappa \leq \mathfrak{c}$ of uncountable cofinality. The following proposition shows that the restriction on the cofinality is really necessary.

Proposition

Let κ be a cardinal of countable cofinality such that $\omega_1 < \kappa < \mathfrak{c}$. Then no subset of 2^ω has the $(\kappa, 2)$-GP.

Some of the "old" results generalize in a straightforward way.

Theorem

- *Assume* MA. *Then 2^ω has the (κ, κ)-GP for every $\kappa < \mathfrak{c}$ of uncountable cofinality.*
- *Assume $\mathfrak{b} = \kappa$. Then 2^ω does not have the (κ, ω_1)-GP.*

In particular, as we have already seen, the statement

"2^ω *has the* $(\mathfrak{c}, \mathfrak{c})$-GP"

is false under CH. Can it be consistently true?

Theorem (Miller, 1983)

It is consistent that for every $Y \subseteq 2^\omega$ *of size* $\mathfrak{c}$ *there exists a continuous function* $f : 2^\omega \longrightarrow 2^\omega$ *such that* $f[Y] = 2^\omega$.

Corollary

It is consistent that 2^ω *has the* $(\mathfrak{c}, \mathfrak{c})$-GP.

The following fundamental question remains open.

Question

For which cardinals κ, λ *such that* $\omega_1 \leq \lambda \leq \kappa \leq \mathfrak{c}$ *and* κ *has uncountable cofinality is it possible to prove in* ZFC *that there exists a* **subset** *of* 2^ω *with the* (κ, λ)-GP*?*

Generalizing the perfect set property

Definition

Fix an uncountable cardinal κ. Let X be a space and Γ a pointclass. We will say that X has the *κ-perfect set property for Γ subsets* (briefly, the κ-PSP(Γ)) if for every Γ subset A of X either $|A| < \kappa$ or A contains a copy of 2^ω.

Notice that the κ-PSP(Γ) gets stronger as κ gets smaller and as Γ gets bigger.
Also, it is clear that the ω_1-PSP(Γ) is simply the PSP(Γ).
As an example, we can rephrase a classical theorem using this terminology.

Theorem (Sierpiński)

Every Polish space has the ω_2-PSP(Σ^1_2).

The Holy Grail

The following is the most general question that we can imagine on this subject.

Question

What is the status of the statement

> *"For every space X, if X has the* κ-PSP(Γ) *then X has the* κ'-PSP(Γ')*"*

as κ, κ' range over all uncountable cardinals and Γ, Γ' range over all pointclasses?

Notice that the "complete picture" can be viewed as a partial answer to the above question, in the case where $\kappa = \kappa' = \omega_1$ and Γ, Γ' are at most analytic.

Question

Can we generalize the "complete picture" from ω_1 to an arbitrary uncountable cardinal κ?

In other words, can we substitute everywhere "PSP(Γ)" with "κ-PSP(Γ)" and "ω_1" with "κ"?
It turns out that the following is the only missing ingredient.

Question

Does $\mathfrak{b} = \kappa$ imply that there exists a space with the κ-PSP(closed) *but not the* κ-PSP(analytic)*?*

The proof that we have given in the case $\kappa = \omega_1$ would go through under the following assumption.

"There exists a subset of 2^ω with the (κ, ω_1)-GP"

We do not know whether the above holds for every κ of uncountable cofinality (or even regular uncountable).

What about PSP(coanalytic)?

Recall that PSP(analytic) $\leftrightarrow$ PSP(G_δ) by Solecki's theorem.
It easily follows that PSP(coanalytic) $\rightarrow$ PSP(analytic).

Proposition (Miller)

There exists a space X with the PSP(analytic) *but not the* PSP(coanalytic)*.*

Proof: Let $\mathrm{WO} \subseteq \omega^\omega$ be the set of codes of well-orderings of ω.
For each infinite $\alpha < \omega_1$, let $z_\alpha \in \mathrm{WO}$ be a code for a well-ordering of ω of type α.
Define $Z = \{z_\alpha : \omega \leq \alpha < \omega_1\}$ and $X = Z \cup (\omega^\omega \setminus \mathrm{WO}) \subseteq \omega^\omega$.
Now apply the following classical results.

- WO is a coanalytic subset of ω^ω.
- If $A \subseteq \mathrm{WO}$ is an analytic subset of ω^ω, then $\{\text{order type of } x : x \in A\}$ is bounded in ω_1.

A new proof of a theorem of Todorčević?

Given an infinite cardinal κ, recall that a subset D of $\mathbb{R}$ is *κ-dense* if $|U \cap D| = \kappa$ for every non-empty open $U \subseteq \mathbb{R}$. Let BA_κ denote the following statement.

> *Whenever D, E are κ-dense subsets of $\mathbb{R}$, there exists a homeomorphism $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $h[D] = E$.*

Theorem (Cantor, 1895)

BA_ω *holds.*

Theorem (Baumgartner, 1984)

Assume PFA*. Then* BA_{ω_1} *holds.*

Theorem (Todorčević, 1988)

Assume $\mathfrak{b} = \kappa$*. Then* BA_κ *does not hold.*

Assume that there exists a subset of 2^ω with the (κ, ω_1)-GP for every regular κ such that $\omega_1 \leq \kappa \leq \mathfrak{c}$. Then we could give a new proof of the result of Todorčević.
Assume that $\kappa = \mathfrak{b}$. The new "proof" would actually give two κ-dense sets which are distinguished by a concrete property.

A κ-dense $D \subseteq \mathbb{R}$ with the (κ, ω_1)-GP.
Fix a base $\{U_n : n \in \omega\}$ for $\mathbb{R}$ and $K_n \subseteq U_n$ for $n \in \omega$ such that $K_n \approx 2^\omega$. Let W_n be a subset of K_n of size κ with the (κ, ω_1)-GP. One can easily check that $D = \bigcup_{n\in\omega} W_n$ is κ-dense in $\mathbb{R}$ and has the (κ, ω_1)-GP.

A κ-dense $E \subseteq \mathbb{R}$ without the (κ, ω_1)-GP.
Fix a closed nowhere dense $K \subseteq \mathbb{R}$ such that $K \approx 2^\omega$. Using the assumption $\kappa = \mathfrak{b}$, let $Z \subseteq K$ be such that $|Z| = \kappa$ and Z does not have the (κ, ω_1)-GP. Extend Z to a κ-dense subset E of $\mathbb{R}$. It is clear that E does not have the (κ, ω_1)-GP.

Thank you for your attention

and good night!