Some uses of homogeneous forcing

David Asperó

University of East Anglia

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A forcing notion $\mathbb{P}$ is *homogeneous* iff for all $p, p' \in \mathbb{P}$ there are $q \leq_{\mathbb{P}} p$ and $q' \leq_{\mathbb{P}} p'$ such that

$$\mathbb{P} \upharpoonright q \cong \mathbb{P} \upharpoonright q'$$

Standard fact: If $\mathbb{P}$ is a homogeneous forcing notion, then for all $p, p' \in \mathbb{P}$ and all statements φ in the forcing language for $\mathbb{P}$ with parameters from the ground model,

$$p \Vdash_{\mathbb{P}} \varphi$$

iff

$$p' \Vdash_{\mathbb{P}} \varphi$$

Hilbert's programme revisited

Some local notation: Given a theory Σ and a sentence σ, in the language of set theory, σ is a Φ–consequence from Σ, denoted

$$\Sigma \vdash_{\Phi} \sigma,$$

iff for every set–forcing $\mathcal{P}$, if $\mathcal{P}$ forces every sentence in Σ, then $\mathcal{P}$ forces σ.

Φ is for 'forcing'.

This definition of course makes sense for choices of Σ for which this can be expressed. For choices of Σ where its members have unbounded Lévy complexity this might of course not be definable. Also, note that the definition makes sense also for choices of Σ which are not even definable (as long as they are in $\mathbf{V}$).

This gives a notion of logic $\models_\Phi$, possibly weaker than the logic $\models_{GM}$ of the generic multiverse.

We use Φ–true, Φ–satisfiable, Φ–complete and so on, in the natural intended way. For example, a theory Σ is Φ–complete for a set Δ of sentences if and
only if for every $\sigma \in \Delta$ at least one of $\Gamma \models_\Phi \sigma$ and $\Gamma \models_\Phi \neg\sigma$ holds.

The usual (Woodin's) definition of Ω–logic can be phrased in the above language, at least for (say) choices of Σ which are definable over ω: Suppose Σ is definable over ω. Then σ is an Ω–consequence of Σ if and only if the sentence "for all ordinals α, if $V_\alpha \models \psi$ for every $\psi \in \Sigma$, then $V_\alpha \models \sigma$" is a Φ–truth (where of course the mention of Σ refers to the definition of Σ).

We may also define relativized versions Φ^Γ of Φ–logic for definable classes Γ of posets.

For example T *is* Φ^Γ*–complete for* Δ iff for every $\sigma \in \Delta$ it holds that either

- for every $\mathbb{P} \in \Gamma$, if $\Vdash_{\mathbb{P}} \varphi$ for every $\varphi \in T$, then $\Vdash_{\mathbb{P}} \sigma$, or
- for every $\mathbb{P} \in \Gamma$, if $\Vdash_{\mathbb{P}} \varphi$ for every $\varphi \in T$, then $\Vdash_{\mathbb{P}} \neg\sigma$.

Σ_2 theories

For Σ_2 theories, i.e., theories of the form $(\exists\alpha)(V_\alpha \models T)$ (equivalently, of the form $(\exists\kappa)(H(\kappa) \models T)$) and Σ_2 sentences σ, Φ–logic coincides with Ω–logic:

$$T \models_\Phi \sigma \text{ iff } T \models_\Omega \sigma$$

Woodin: If there is a proper class of Woodin and the Ω Conjecture is true, then:

1. the $\mathbb{P}_{max}$–axiom $(*)$ is Ω–satisfiable (equiv., it can always be obtained by set–forcing over any set–forcing extension). Hence, since $(*)$ is Φ–complete for $\mathrm{Th}(H(\omega_2))$, if the Ω Conjecture is true under every large cardinal hypothesis, then $(*)$ is an axiom which is
   - compatible with all large cardinals,
   - Ω–complete for $\mathrm{Th}(H(\omega_2))$, and
   - which can always be set–forced after any set–forcing.
2. There is no Ω–satisfiable theory which is Ω–complete for $\mathrm{Th}(H(\delta_0^+))$, where δ_0 is the least Woodin cardinal.

Woodin: If there is a proper class of Woodin and the Strong Ω Conjecture is true, then:

1. The Ω Conjecture is true.
2. All theories which are Ω–complete for $\mathrm{Th}((H(\omega_2))$ imply $\neg\,\mathrm{CH}$.
3. There is no Ω–satisfiable theory which is Ω–complete for the Σ^2_3 theory.

In “Incompatible Ω–complete theories”, JSL 2009, Koellner and Woodin contemplate the following very optimistic scenario:

Could it be, in a large cardinal context, that the following holds?

(i) The Ω Conjecture is false.

(ii) There is a sequence of Ω–satisfiable Σ_2 theories which are Ω–complete for the theory of larger and larger (all ?) reasonably specifiable initial segments of the universe.

(iii) All these theories give the same theory of the relevant initial segments of the universe.

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Koellner and Woodin show that if (i) and (ii) hold, then (iii) has to fail (granting liberal use of large cardinals, as usual).

They show that if there is a Σ_2 theory T which, modulo some large cardinal assumption LC, is Ω–satisfiable and Ω–complete for (say) $\mathrm{Th}(H(\kappa))$, for $\kappa = (2^{\aleph_0})^+$, then there are Σ_2 theories T^{CH}, $T^{\neg\,\mathrm{CH}}$ which, modulo slightly stronger large cardinal assumption LC′, are Ω–satisfiable and Ω–complete for $\mathrm{Th}(H(\omega_2))$ and such that

- $T^{\mathrm{CH}} \vdash \mathrm{CH}$ and
- $T^{\neg\,\mathrm{CH}} \vdash \neg\,\mathrm{CH}$.

Proof proceeds by considering the theories that (essentially) say

- “I am a forcing extension of a model of T by $\mathrm{Add}(\omega_1, 1)$” (for T^{CH})
- “I am a forcing extension of a model of T by $\mathrm{Add}(\omega, \omega_2)$” (for $T^{\neg\mathrm{CH}}$)

The main points are:

- $\mathrm{Add}(\omega_1, 1)$ and $\mathrm{Add}(\omega, \omega_2)$ are definable over $H(\kappa)$ from no parameters and homogeneous.
- κ is large enough that all nice names for members of $H(\omega_2)$ are in $H(\kappa)$.

CH, $\neg$CH is clearly not the only pair they can deal with. A similar result can be proved for any Σ_2 statement σ such that both σ and $\neg\sigma$ can be forced by some similarly nice forcing.

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Down to $H(\omega_2)$

Consider the question:

Question: Does the existence of an Ω–satisfiable Σ_2–theory T which is Ω–complete for $\mathrm{Th}(H(\omega_2))$ imply the existence of another such theory incompatible with T?

[Koellner–Woodin] does not address this question: their use of $\mathrm{Add}(\omega, \omega_2)$ does address the problem of producing a theory implying $\neg\,\mathrm{CH}$, but $\mathrm{Add}(\omega_1, 1)$ is not suitable for building a theory implying CH (in our context): If CH fails, then there are nice $\mathrm{Add}(\omega_1, 1)$–names for members of $H(\omega_2)$ which are not in $H(\omega_2)$. In fact $\mathrm{Add}(\omega_1, 1)$ will collapse ω_2.

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Addressing the question

Plan: Use [Koellner–Woodin]'s result in the following form:

Theorem [Koellner–Woodin] Suppose there is a proper class of Woodin cardinals. Suppose φ is a Σ_2 large cardinal property and ψ is a Σ_2 sentence such that T= ZFC + ψ + "There is a proper class of Wodin cardinals" + "There is a proper class of φ–cardinals" is Ω–complete for $\mathrm{Th}(H(\omega_2))$. Let $\mathbb{P} \subseteq H(\omega_2)$ be a forcing such that T Ω–implies that

(1) $\mathbb{P}$ is definable over $H(\omega_2)$ (from no parameters).

(2) $\mathbb{P}$ is homogeneous.

(3) $\mathbb{P}$ preserves ω_1 and has the $\aleph_2$–c.c. (in particular every $\mathbb{P}$–name for a member of $H(\omega_2)$ can be assumed to be in $H(\omega_2)$).

Let $T_{\mathbb{P}}$ be the sentence:

There is (κ, N, G) such that

- κ is an inaccessible cardinal,
- $N \models T$,
- G is $\mathbb{P}^N$–generic over $H(\omega_2)^N$, and
- $H(\omega_2) = H(\omega_2)^{N[G]}$.

Then the sentence

ZFC+ $T_{\mathbb{P}}$ +"There is a proper class of Wodin cardinals" + "There is a proper class of φ–cardinals"

is Ω–complete for $\text{Th}(H(\omega_2))$.

A definable homogeneous version of the Hechler iteration

Goal: Want to force $\mathfrak{b} > \omega_1$ by a forcing $\mathbb{P} \subseteq H(\omega_2)$ such that:

(1) $\mathbb{P}$ is definable over $H(\omega_2)$ (from no parameters).

(2) $\mathbb{P}$ is homogeneous.

(3) $\mathbb{P}$ preserves ω_1 and has the $\aleph_2$–c.c. (in particular every $\mathbb{P}$–name for a member of $H(\omega_2)$ can be assumed to be in $H(\omega_2)$).

(4) $\mathbb{P}$ forces $\mathfrak{b} > \omega_1$.

First approximation: Consider the following Hechler iteration: $\mathbb{P} = \mathbb{P}_{\omega_2}$, where $(\mathbb{P}_\alpha)_{\alpha \leq \omega_2}$ is such that for all α:

(a) Conditions in $\mathbb{P}_\alpha$ are finite functions $p = ((s_\beta, \mathcal{F}_\beta) : \beta \in \mathrm{dom}(p))$ such that $\mathrm{dom}(p) \subseteq \alpha$ and for all $\beta \in \mathrm{dom}(p)$:
  - $s_\beta \in {}^{<\omega}\omega$
  - $\mathcal{F}_\beta$ is a finite set of names (in $H(\omega_2)$) $\dot{f}$ such that $\Vdash_{\mathbb{P}_\beta} \dot{f} \in {}^{\omega}\omega$.

(b) Given $p^0 = ((s^0_\alpha, \mathcal{F}^0_\beta) : \beta \in \mathrm{dom}(p^0))$, $p^1 = ((s^1_\beta, \mathcal{F}^1_\beta) : \beta \in \mathrm{dom}(p^1)) \in \mathbb{P}_\alpha$, p^1 extends p^0 iff $\mathrm{dom}(p^0) \subseteq \mathrm{dom}(p^1)$ and for all $\beta \in \mathrm{dom}(p^0)$,
  - s^1_β extends s^0_β,
  - $\mathcal{F}^0_\beta \subseteq \mathcal{F}^1_\beta$, and
  - for every $\dot{f} \in \mathcal{F}^0_\beta$ and every $k \in |s^1_\beta| \setminus |s^0_\beta|$, $p^1 \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{f}(k) < s^1_\beta(k)$.

$\mathbb{P} = \mathbb{P}_{\omega_2}$ forces $\mathfrak{b} > \omega_1$, it preserves ω_1 and has the $\aleph_2$–c.c. (in fact it has the c.c.c.), and it is homogeneous.

On the other hand $\mathbb{P}$ does not seem to be definable over $H(\omega_2)$:

The reference, in the definition of $\mathbb{P}_\beta$, to arbitrary $\mathbb{P}_\alpha$–names, for $\alpha < \beta$, blows up the complexity of the definition. It is not clear that even $\mathbb{P}_\omega$ is definable over $H(\omega_2)$.

How to ensure definability?

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How to ensure definability?

Mimicking forcing iterations by definable homogeneous forcing

Notation: Given functions p, p' with ranges consisting of ordered pairs, *p and p' are compatible* iff for all $x \in \mathrm{dom}(p) \cap \mathrm{dom}(p')$, $p(x) = (Y, \tau)$ and $p'(x) = (Y', \tau)$ for some Y, Y' and τ.

Given functions p, p' with ranges consisting of ordered pairs, if p and p' are compatible, then

$$p \wedge p'$$

denotes the function with $\mathrm{dom}(p \wedge p') = \mathrm{dom}(p) \cup \mathrm{dom}(p')$ such that

- $(p \wedge p') \restriction \mathrm{dom}(p) \setminus \mathrm{dom}(p') = p \restriction \mathrm{dom}(p) \setminus \mathrm{dom}(p')$,
- $(p \wedge p') \restriction \mathrm{dom}(p') \setminus \mathrm{dom}(p) = p' \restriction \mathrm{dom}(p') \setminus \mathrm{dom}(p)$, and
- for all $x \in \mathrm{dom}(p) \cap \mathrm{dom}(p')$, if $p(x) = (Y, \tau)$ and $p'(x) = (Y', \tau)$, then $(p \wedge p')(x) = (Y \cup Y', \tau)$.

Let $\mathcal{P}^{\mathfrak{b}>\omega_1}$ be the following poset: $q \in \mathcal{P}^{\mathfrak{b}>\omega_1}$ iff q is a finite function consisting of pairs $((\alpha, \mathbb{P}, \dot{\mathcal{F}}), (X, \sigma))$ where:

(1) $\alpha \in \omega_2$

(2) $\mathbb{P} \in H(\omega_2)$ is a partial order consisting of finite sets of pairs $((\beta, \mathbb{Q}, \dot{\mathcal{G}}), (Y, \tau))$, where

- $\beta \in \omega_2$
- $\mathbb{Q}, \dot{\mathcal{G}}, Y, \in H(\omega_2)$, $\tau \in {}^{<\omega}\omega$, and such that
- for all p, $p' \in \mathbb{P}$, if p and p' are compatible, then $p \wedge p'$ is a common extension of p and p' in $\mathbb{P}$.

(3) $\dot{\mathcal{F}} \in H(\omega_2)$ is a $\mathbb{P}$–name for an ω_1–sequence of members of ${}^{\omega}\omega$.

(4) $X \in [\omega_1]^{<\omega}$

(5) $\sigma \in {}^{<\omega}\omega$

Given $q_0, q_1 \in \mathcal{P}^{\mathfrak{b}>\omega_1}$, q_1 extends q_0 iff

(a) $\mathrm{dom}(q_0) \subseteq \mathrm{dom}(q_1)$

(b) For every $(\alpha, \mathbb{P}, \mathcal{F}) \in \mathrm{dom}(q_0)$, if $q_0((\alpha, \mathbb{P}, \mathcal{F})) = (X_0, \sigma_0)$ and $q_1((\alpha, \mathbb{P}, \mathcal{F})) = (X_1, \sigma_1)$, then

   (i) $X_0 \subseteq X_1$,

   (ii) $\sigma_0 \subseteq \sigma_1$, and

   (iii) for all $\nu \in X_0$ and all $n \in \mathrm{dom}(\sigma_1) \setminus \mathrm{dom}(\sigma_0)$, $q_1 \cap \mathbb{P} \in \mathbb{P}$ and there is some $s \in {}^{|\sigma_1|}\omega$ such that

$$q_1 \cap \mathbb{P} \Vdash_{\mathbb{P}} \dot{\mathcal{F}}(\nu) \restriction |\sigma_1| = \check{s}$$

   and such that

$$\sigma_1(n) > s(n)$$

Proposition

$\mathcal{P}^{\mathfrak{b}>\omega_1}$ has the following properties.

1. *It is definable over $H(\omega_2)$ from no parameters.*
2. *It is homogeneous.*
3. *It has precalibre $\aleph_1$ (i.e., every uncountable collection of conditions includes an uncountable set X such that for every $x \in [X]^{<\omega}$ there is a common lower bound for all conditions in x).*
4. *It forces $\mathfrak{b} > \omega_1$.*

We immediately get the following.

Theorem

Suppose there is, under some sufficiently strong large cardinal assumption LC*, a recursively enumerable* Ω*–satisfiable* Σ_2*–theory* T *such that* T *is* Ω*–complete for* $\mathrm{Th}(H(\omega_2))$*. Then there are, under a slightly stronger large cardinal assumption* LC'*,* Ω*–satisfiable recursively enumerable* Σ_2*–theories* $T^{\mathfrak{b}=\omega_1}$ *and* $T^{\mathfrak{b}>\omega_1}$ *such that*

1. $T^{\mathfrak{b}=\omega_1}$ *and* $T^{\mathfrak{b}>\omega_1}$ *are both* Ω*–complete for the theory of* $H(\omega_2)$*.*
2. $T^{\mathfrak{b}=\omega_1} \vdash \mathfrak{b} = \omega_1$
3. $T^{\mathfrak{b}>\omega_1} \vdash \mathfrak{b} > \omega_1$

Proof: By the above proposition together with an application of the Koellner–Woodin argument with $\mathrm{Add}(\omega, \omega_2)$ (for $T^{\mathfrak{b}=\omega_1}$) and with $\mathcal{P}^{\mathfrak{b}>\omega_1}$ (for $T^{\mathfrak{b}>\omega_1}$). $\square$

Another way to do this: Weak forms of Club Guessing and their negations

Club Guessing at ω_1 (CG): There is a ladder system $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ such that for every club $C \subseteq \omega_1$ there is some δ such that $C_\delta \setminus C$ is finite.

Interval Hitting Principle (IHP):

There is a ladder system $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ such that for every club $C \subseteq \omega_1$ there is some δ such that

$$[C_\delta(n), C_\delta(n+1)) \cap C \neq \emptyset$$

for a tail of $n < \omega$.

Here, $(C_\delta(n))_{n<\omega}$ is the strictly increasing enumeration of C_δ. (IHP is due to Kunen and is sometimes called Kunen's Axiom.)

Consider the following forcing $\mathcal{P}^{\mathrm{IHP}}$.

Definition

Conditions in $\mathcal{P}^{\mathrm{IHP}}$ are pairs $q = (\vec{c}, \vec{D})$ with the following properties.

- $\vec{c} = (c_\delta : \delta \in S)$ is a finite function with $S \subseteq Lim(\omega_1)$ and such that for every $\delta \in S$, $c_\delta \subseteq \omega \times \delta$ is a finite strictly increasing function.
- $\vec{D} = (\mathcal{D}_\delta : \delta \in T)$ is such that $T \subseteq Lim(\omega_1)$ is finite and for every δ, $\mathcal{D}$ is a finite set of cofinal subsets of δ of order type ω.

Given $(\vec{c}^0, \vec{D}^0), (\vec{c}^1, \vec{D}^1) \in \mathcal{P}^{\mathrm{IHP}}$, $(\vec{c}^1, \vec{D}^1) \leq (\vec{c}^0, \vec{D}^0)$ iff:

(1) $\mathrm{dom}(\vec{c}^0) \subseteq \mathrm{dom}(\vec{c}^1)$ and $c^0_\delta \subseteq c^1_\delta$ for every $\delta \in \mathrm{dom}(\vec{c}^0)$.

(2) For every $\mathrm{dom}(\vec{D}^0) \subseteq \mathrm{dom}(\vec{D}^1)$ and every $\delta \in \mathrm{dom}(\vec{D}^0)$, $\mathcal{D}^0_\delta \subseteq \mathcal{D}^1_\delta$.

(3) For every $\delta \in \mathrm{dom}(\vec{c}^0)$ and n, $n+1 \in \mathrm{dom}(c^1_\delta) \setminus \mathrm{dom}(c^0_\delta)$, if $\delta \in \mathrm{dom}(\vec{D}^0)$, then $[c^1_\delta(n), c^1_\delta(n+1)) \cap D \neq \emptyset$ for every $D \in \mathcal{D}^0_\delta$.

Proposition

$\mathcal{P}^{IHP}$ *has the following properties.*

1. *It is definable over* $H(\omega_2)$ *from no parameters.*
2. *It is homogeneous.*
3. *It has the c.c.c.*
4. *It forces* IHP.

Let $\mathbb{B}$ denote Baumgartner's forcing for adding a club of ω_1 with finite conditions. Adding many Baumgartner clubs of ω_1:

Given a set X of ordinals, there is a forcing, which I will denote by $\text{Add}_{\mathbb{B}}(X)$, with the following properties.

(1) For every $\text{Add}_{\mathbb{B}}(X)$–generic G and every $\alpha \in X$ one can naturally extract a Baumgartner club C^G_α from G. Moreover, $C^G_\alpha \neq C^G_{\alpha'}$ for $\alpha \neq \alpha'$ in X.

(2) $\text{Add}_{\mathbb{B}}(X)$ is proper and has the $\aleph_2$–c.c.

(3) For every partition (X_0, X_1) of X into nonempty pieces, $\text{Add}_{\mathbb{B}}(X) \cong \text{Add}_{\mathbb{B}}(X_0) \times \text{Add}_{\mathbb{B}}(X_1)$. In particular, if G is $\text{Add}_{\mathbb{B}}(X)$–generic and $\alpha \neq \alpha'$ are in X, then C^G_α is $\mathbb{B}$–generic over $\mathbf{V}[C^G_{\alpha'}]$.

(4) $\text{Add}_{\mathbb{B}}(X)$ is homogeneous.

It follows from (1), (2) and (3) that if $\text{ot}(X) \geq \omega_2$, then $\text{Add}_{\mathbb{B}}(X)$ forces $\neg$ IHP.

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It follows from (1), (2) and (3) that if $\mathrm{ot}(X) \geq \omega_2$, then $\mathrm{Add}_{\mathbb{B}}(X)$ forces $\neg\,\mathrm{IHP}$.

Defininition: Let X be a set of ordinals. $\mathrm{Add}_{\mathbb{B}}(X)$ is the following forcing: Conditions in $\mathrm{Add}_{\mathbb{B}}(X)$ are pairs of the form $p = (f, \mathcal{F})$ with the following properties.

(1) f is a finite function with $\mathrm{dom}(f) \subseteq X$ and such that $f(\alpha) \in \mathbb{B}$ for every $\alpha \in \mathrm{dom}(f)$.

(2) $\mathcal{F}$ is a finite function with $\mathrm{dom}(\mathcal{F}) \subseteq \omega_1$ such that for every $\delta \in \mathrm{dom}(\mathcal{F})$,
   (a) δ is a countable indecomposable ordinal,
   (b) $\mathcal{F}(\delta)$ is a countable subset of X,
   (c) $\delta \in \mathrm{dom}(f(\alpha))$ and $f(\alpha)(\delta) = \delta$ for all $\alpha \in \mathrm{dom}(f) \cap \mathcal{F}(\delta)$, and
   (d) for every $\delta' \in \mathrm{dom}(\mathcal{F} \restriction \delta)$ and every $\alpha \in F(\delta)$, $\mathrm{rank}(\mathcal{F}(\delta'), \alpha) < \delta$.

Given $(f_0, \mathcal{F}_0), (f_1, \mathcal{F}_1) \in \mathrm{Add}_{\mathbb{B}}(X)$, $(f_1, \mathcal{F}_1)$ extends $(f_0, \mathcal{F}_0)$ iff

- $\mathrm{dom}(f_0) \subseteq \mathrm{dom}(f_1)$ and $f_0(\alpha) \subseteq f_1(\alpha)$ for every $\alpha \in \mathrm{dom}(f_0)$, and
- $\mathrm{dom}(\mathcal{F}_0) \subseteq \mathrm{dom}(\mathcal{F}_1)$ and $\mathcal{F}_0(\delta) \subseteq \mathcal{F}_1(\delta)$ for every $\delta \in \mathrm{dom}(\mathcal{F}_0)$.

We immediately get the following.

Theorem

Suppose there is, under some sufficiently strong large cardinal assumption LC, a recursively enumerable Ω–satisfiable Σ_2–theory T such that T is Ω–complete for $\mathrm{Th}(H(\omega_2))$. Then there are, under a slightly stronger large cardinal assumption LC', Ω–satisfiable recursively enumerable Σ_2–theories T^{IHP} and $T^{\neg IHP}$ such that

1. T^{IHP} *and* $T^{\neg IHP}$ *are both Ω–complete for the theory of* $H(\omega_2)$.
2. $T^{IHP} \vdash IHP$
3. $T^{\neg IHP} \vdash \neg IHP$

Proof: By the above together with an application of the Koellner–Woodin argument with $\mathcal{P}^{\mathrm{IHP}}$ (for T^{IHP}) and with $\mathrm{Add}_{\mathbb{B}}(\omega_2)$ (for $T^{\neg\,\mathrm{IHP}}$). $\square$

Stronger results involving Club–Guessing

Weak Club Guessing (WCG): There is a ladder system $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ such that for every club $C \subseteq \omega_1$ there is some δ such that $C_\delta \cap C$ is infinite.

It seems there are also forcing notions $\mathcal{P}^{\neg\,\mathrm{WCG}}$, $\mathcal{P}^{\mathrm{CG}}$ such that

1. $\mathcal{P}^{\neg\,\mathrm{WCG}}$ and $\mathcal{P}^{\mathrm{CG}}$ are definable over $H(\omega_2)$ from no parameters.
2. $\mathcal{P}^{\neg\,\mathrm{WCG}}$ and $\mathcal{P}^{\mathrm{CG}}$ are homogeneous.
3. $\mathcal{P}^{\neg\,\mathrm{WCG}}$ and $\mathcal{P}^{\mathrm{CG}}$ are proper and have the $\aleph_2$–c.c.
4. $\mathcal{P}^{\neg\,\mathrm{WCG}}$ forces $\neg\,\mathrm{WCG}$ and $\neg\,\mathrm{IHP}$.
5. $\mathcal{P}^{\mathrm{CG}}$ forces CG.

There is of course a corresponding corollary mentioning theories, Ω–complete for the theory of $H(\omega_2)$, $T^{\neg\,\mathrm{WCG}+\neg\,\mathrm{IHP}}$ and T^{CG}.

$\mathcal{P}^{\neg\,\mathsf{WCG}}$ and $\mathcal{P}^{\mathsf{CG}}$

- ‘mimick’ forcing iterations (like in the definition of $\mathcal{P}^{\mathfrak{b}>\omega_1}$) and also
- incorporate side conditions (like in the definition of $\mathcal{P}^{\mathsf{IHP}}$ and of $\mathrm{Add}_B(X)$).

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This is still work under construction.

Many natural questions spring from here. For example:

- In the presence of some reasonable sufficiently strong large cardinal axiom. Do all Ω–satisfiable recursive Σ_2–theories which are Ω–complete for the theory of $H(\omega_2)$ imply the existence of a Suslin tree?

  (If yes, then of course the $\mathbb{P}_{max}$ axiom $(*)$ could not be Ω–satisfiable.)

- Do all Ω–satisfiable recursive Σ_2–theories which are Ω–complete for the theory of $H(\omega_2)$ imply $\neg\,\mathrm{CH}$?

Further advertising $\mathrm{Add}_{\mathbb{B}}(X)$: Collapsing exactly $\aleph_3$

$\mathrm{Add}_{\mathbb{B}}(X)$ has other interesting uses. Here is an example:

U. Abraham proves the following in *On forcing without the continuum hypothesis*, J. Symbolic Logic, vol. 48, 3 (1983), 658–661:

Theorem

(Abraham) (ZFC) There is a poset $\mathcal{P}$ collapsing ω_2 and preserving all other cardinals.

Abraham's forcing is built as follows: Let $A \subseteq \omega_2$ such that $\omega_2^{L[A]} = \omega_2^{\mathbf{V}}$ (and then of course $\omega_1^{L[A]} = \omega_1^{\mathbf{V}}$). Then

$$\mathcal{P} = \mathrm{Add}(\omega, \omega_1) * \mathrm{Coll}(\omega_1, \omega_2)^{LA][\dot{G}]}$$

- $\mathcal{P}$ collapses ω_2 and has a dense subset of size $\aleph_2$.
- Preservation of ω_1: If G is $\mathrm{Add}(\omega, \omega_1)$–generic, $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][G]}$ is σ–closed in $L[A][G]$, but certainly not in general in $\mathbf{V}[G]$. However, $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][G]}$ is σ–distribuitive in $\mathbf{V}[G]$:

  Given a $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$–condition p and a $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][G]}$–name $\dot{F}$ in $\mathbf{V}[G]$ for a function $\dot{F} : \omega \longrightarrow \mathrm{Ord}$, we may find a condition $p' \leq p$ in $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][G]}$ deciding all of $\dot{F}$. We use the Cohen reals added by G in order to guide this construction (in $\mathbf{V}[G]$).

Question

(in Abraham's paper) Can this be extended to higher cardinals? In particular, is there, in ZFC*, a forcing collapsing exactly* $\aleph_3$*?*

Theorem

(ZFC) There is a poset $\mathcal{P}$ *collapsing* $\aleph_3$ *and preserving all other cardinals.*

Construction of $\mathcal{P}$: There is a poset $\mathcal{P}_0$ of size $\aleph_2$ preserving cardinals and adding a partial $\square_{\omega_1}$–sequence $(C_\alpha : \alpha \in S)$ such that $\{\alpha \in S : \mathrm{cf}(\alpha) = \omega_1\}$ is stationary. In $\mathbf{V}_1 = \mathbf{V}^{\mathcal{P}_0}$ we may then fix $A \subseteq \omega_3$ such that $\omega_3^{L[A]} = \omega_3$ and such that for every cardinal $\theta > \omega_3$, the set of $N \prec H(\theta)$ such that

- $|N| = \aleph_1$,
- $N \cap H(\omega_3)^{L[A]} \in L[A]$, and
- $N \cap H(\omega_3)^{L[A]}$ is internally approachable in $L[A]$

is a stationary subset of $[H(\theta)]^{\aleph_1}$. Still in $\mathbf{V}_1$, let $\mathcal{P}_1 = \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]} * \dot{\mathcal{Q}}$, where $\dot{\mathcal{Q}}$ is, in $L[A]^{\mathrm{Add}_{\mathbb{B}}(\omega_1)}$, a name for $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$. Our poset will be $\mathcal{P} = \mathcal{P}_0 * \dot{\mathcal{P}}_1$, where $\dot{\mathcal{P}}_1$ is a $\mathcal{P}_0$–name for $\mathcal{P}_1$. $\square$

Question: Is there, in ZFC, a forcing notion collapsing $\aleph_4$ and preserving all other cardinals? What about for *any* $\kappa \neq \omega_1, \omega_2, \omega_3$?

Relative definability

We all know that in ZFC one can prove the existence of such objects as Hausdorff gaps, Aronszajn trees, partitions of ω_1 into $\aleph_1$–many stationary sets, and so on. Many of these existence proofs proceed by a specific construction of the relevant type of object, where this construction is definable from any given object p satisfying a certain property P: One establishes in ZFC the existence of some p such that $P(p)$, and then one runs the relevant construction with any fixed p such that $P(p)$ as a parameter.

Typical example: If $\vec{C}$ is a ladder system, then there is a recursive construction of a Countryman line definable from $\vec{C}$ (Todorčević).

I will look next at the question: "if A is such that $P(A)$, does there exists a B such that $Q(B)$ and B is definable from A?" for various classical properties $P(x)$, $Q(x)$ of combinatorial flavour pertaining the structure $H(\omega_2)$.

Some positive results

Given two properties $P(x)$, $Q(x)$, I will say that $P(x)$ *has definability strength at least that of* $Q(x)$ *over* $\langle H(\omega_2), \in\rangle$ if there is a formula $\varphi(x, y)$ such that
$Q(\{b \in H(\omega_2) : H(\omega_2) \models \varphi(A, b)\})$ for every $A \in H(\omega_2)$ such that $P(A)$.

Proposition

(ZF) The following properties have the same definability strength over $\langle H(\omega_2), \in\rangle$.

1. x *is a ladder system*
2. x *is a simplified* $(\omega, 1)$*–morass*
3. x *is an special Aronszajn tree with a witness*
4. x *is a Countryman line with a witness*
5. x *is an indestructible gap with a witness*

Sample proof: If $p = (\leq_C, (X_n)_{n\in\omega})$ is a Countryman line with a witness, then there is a ladder system definable from p:

Let $\theta = \omega_2$, and let $A \subseteq \omega_1$ be defined from p in $H(\omega_2)$ and such that $L_\theta(p) = L_\theta(A) = L_\theta[A]$. If $\kappa = \omega_1$, then in $L_\theta[A]$, $p = ((\kappa, \leq_C), (X_n)_{n\in\omega})$, where $\leq_C$ is a linear order on κ and $(X_n)_{n\in\omega}$ is a decomposition of $\kappa \times \kappa$ into chains. But then necessarily $\kappa = \omega_1^{L[A]}$:
$L_\theta[A]$ can see that $(\kappa, \leq_C)$ embeds neither what it thinks is ω_1, nor its converse, nor any uncountable set of reals (not difficult to verify and first observed by Galvin), and therefore it believes (correctly) that $|\kappa| = \aleph_1$:
$L_\theta[A]$ sees that any partition tree for $(\kappa, \leq_C)$ has to be an Aronszajn tree on its ω_1, and therefore it sees also $|\kappa| = \aleph_1$. But then $\kappa = \omega_1^{L_\theta[A]}$. Now we can pick the $<_{L_\theta[A]}$–first ladder system on ω_1 in $L_\theta[A]$. $\square$

Some negative results

Theorem

1. *It is consistent that there is an Aronszajn tree* T*, an* (ω_1, ω_1)*–gap* $(\vec{f}, \vec{g})$ *in* $({}^\omega\omega, <^*)$ *and a partition* $\vec{S}$ *of* ω_1 *into* $\aleph_1$*–many stationary sets such that no ladder system is definable from* $(T, (\vec{f}, \vec{g}), \vec{S})$*.*
2. *If there is an inaccessible cardinal, then the following holds in a symmetric submodel of a forcing extension of* V*: There is an Aronszajn tree, an* (ω_1, ω_1)*–gap in* $({}^\omega\omega, <^*)$ *and a partition of* ω_1 *into* $\aleph_1$*–many stationary sets but there is no ladder system on* ω_1*.*

Proof of (1): Start with a model with an $\aleph_2$–Aronszajn tree T. Let $\vec{S} = (S_\nu)_{\nu<\omega_2}$ be any partition of ω_2 into stationary sets. Let $\mathcal{P}$ be c.c.c. forcing for adding (ω_1, ω_1)–gap, let G be $\mathcal{P}$–generic and let $(\vec{f}, \vec{g})$ be the generic gap added by G.

Claim

Every c.c.c. forcing $\mathcal{Q}$ preserves the Aronszajnness of T. In particular, T is Aronszajn in $V[G]$.

Proof.

Otherwise there is a $\mathcal{Q}$–name $\dot{b}$ for a cofinal branch through T and a subtree $T' \subseteq T$ of height ω_2 with countable levels such that $\Vdash_{\mathcal{Q}} \dot{b} \restriction \alpha \in T'_\alpha$ for every $\alpha < \omega_2$. But for every regular κ and $\lambda < \kappa$, every tree of height κ with levels of size less than λ has a κ–branch, and so T', and therefore also T, has an ω_2–branch, which is a contradiction. □

In $V[G]^{\mathrm{Coll}(\omega,\omega_1)}$, every S_ν remains a stationary subset of $\omega_2^{V[G]} = \omega_2^V$, $\omega_1 = \omega_2^V$, and $(\vec{f}, \vec{g})$ is still a gap: Suppose G' is $\mathrm{Coll}(\omega, \omega_1)$–generic over $V[G]$ and r is a real in $V[G][G']$. Then $r \in V[G \restriction \alpha][G']$ for some $\alpha < \omega_2$. But then r cannot split $(\vec{f}, \vec{g})$.

Finally, in $V[G]^{\mathrm{Coll}(\omega,\omega_1)}$ there cannot be any ladder system on $\omega_1 = \omega_2^{V[G]}$ definable from $(T, (\vec{f}, \vec{g}), \vec{S})$. Otherwise, by homogeneity of the collapse this ladder system would be in $V[G]$, which is impossible.

Proof of (2): Let κ be an inaccessible cardinal such that there is a κ–Aronszajn tree T, let $(S_\nu)_{\nu<\kappa}$ be a partition of κ into stationary sets, and let G be generic for $\mathcal{P}^\kappa_\kappa$. Our model **W** will be the symmetric submodel of the extension of $V[G]$ by $\mathrm{Coll}(\omega, {<}\kappa)$ generated by the names fixed by an automorphism of $\mathrm{Coll}(\omega, {<}\kappa)$ fixing $\mathrm{Coll}(\omega, {<}\alpha)$ for some $\alpha < \kappa$. In **W**, every α is collapsed to ω and so $\omega_1 = \kappa$, each S_ν is clearly stationary, T remains Aronszajn (a cofinal branch through T in **W** would have to be in $V[G][H]$ for a $\mathrm{Coll}(\omega, {<}\alpha)$–generic H for some $\alpha < \kappa$), and $(\vec{f}, \vec{g})$ remains unsplit (by the same proof as in the first part). Also, in **W** there is no ladder system on ω_1 as such an object would be in $V[G][H]$ for some H as above, which is impossible. $\square$

Partitions of ω_1 into stationary sets

Fact

1. *(ZF + the club–filter on ω_1 is normal) If $\vec{C}$ is a C–sequence, then there is a partition of ω_1 into $\aleph_1$–many stationary sets definable from $\vec{C}$.*
2. *(ZF $+$ DC) If $\vec{r} = (r_\alpha)_{\alpha<\omega_1}$ is a one–to–one ω_1–sequence of reals, then there is a partition of ω_1 into $\aleph_0$–many stationary sets definable from $\vec{r}$.*

I don't know if the normality of the club–filter is need in the first part and if DC is needed in the second part. In fact I don't even know whether ZF alone implies that if $\vec{r}$ is a one–to–one ω_1–sequence of reals, then there is a stationary and co–stationary subset of ω_1 definable from $\vec{r}$.

Theorem

Let $\lambda \leq \omega$ be a nonzero cardinal. The following theories are equiconsistent.

1. ZFC + *There is a measurable cardinal.*
2. ZFC + *There is a partition $(S_i)_{i<\lambda}$ of ω_1 into stationary sets such that no partition of ω_1 into more than λ–many stationary sets is definable from $(S_i)_{i<\lambda}$.*

Proof: Let κ be measurable. By a classical result of Kunen–Paris we may assume that there are distinct normal measures $\mathcal{U}_i$ on κ for $i < \lambda$. We may then find stationary subsets S_i of κ, for $i < \lambda$, such that for all $i^* < \lambda$, i^* is the unique $i < \lambda$ such that $S_{i^*} \in \mathcal{U}_i$. We may assume that each S_i consists of inaccessible cardinals.

In $\mathbf{V}^{\mathrm{Coll}(\omega, <\kappa)}$, let $\dot{\mathcal{P}}$ be a homogeneous forcing preserving the stationarity of all S_i and adding a club C of $\kappa = \omega_1$, $C \subseteq \bigcup_{i<\lambda} S_i$, together with enumerations $(X^i_\alpha)_{\alpha<\kappa}$ of $\mathcal{U}_i$ for each $i < \lambda$ such that for all $\alpha \in C \cap S_i$, $\alpha \in \bigcap_{\beta<\alpha} X^i_\beta$.

Let H be $\mathrm{Coll}(\omega, {<}\kappa) * \dot{\mathcal{P}}$–generic over V, let C be the generic club of κ added by $\dot{\mathcal{P}}$ over $V^{\mathrm{Coll}(\omega, {<}\kappa)}$, and suppose, towards a contradiction, that there is a cardinal $\lambda' > \lambda$ and a partition $(A_i)_{i<\lambda'}$ of $\omega_1^{V[H]} = \kappa$ into stationary sets definable from $(S_i)_{i<\lambda}$. By homogeneity of $\mathrm{Coll}(\omega, {<}\kappa) * \dot{\mathcal{P}}$, $(A_i)_{i<\lambda'} \in V$. There must then be some $i^* < \lambda$ and two distinct $i_0, i_1 < \lambda'$ such that both $A_{i_0} \cap S_{i^*}$ and $A_{i_1} \cap S_{i^*}$ are stationary in $\mathbf{V}[H]$. There can be at most one $\epsilon \in \{0, 1\}$ such that $A_{i_\epsilon} \cap S_{i^*} \in \mathcal{U}_{i^*}$. In that case it follows that a final segment of $A_{i_\epsilon} \cap S_{i^*}$ is contained in C. But then $A_{i_{1-\epsilon}} \cap S_{i^*}$ is non-stationary, which is a contradiction. And if no $A_{i_\epsilon} \cap S_{i^*}$ is in $\mathcal{U}_{i^*}$, then of course no $A_{i_\epsilon} \cap S_{i^*}$ is stationary, which again is a contradiction.

For the other direction, suppose $(S_i)_{i<\lambda}$ is a partition of ω_1 into stationary sets such that there is no partition of ω_1 into more than λ–many stationary sets definable from $(S_i)_{i<\lambda}$. Let A be a set of ordinals definable from, and coding $(S_i)_{i<\lambda}$. We show that $\kappa = \omega_1$ is measurable in the ZFC–model HOD(A).This is easy if λ is finite; in fact, in this case, for every $i < \lambda$, the club filter on ω_1 restricted to S_i is, in HOD(A), a κ–complete ultrafilter on κ.

If $\lambda = \omega$, fix any $i < \lambda$ and assume towards a contradiction that there is no stationary $S \subseteq S_i$ in $\mathrm{HOD}(A)$ such that the club filter on κ is an ultrafilter in $\mathrm{HOD}(A)$. Then we can define from A a $\subseteq$–maximal assignment $(S_t \,:\, t \in T)$ of stationary subsets of S_i, for some tree $T \subseteq {}^{<\kappa}2$, such that $S_{t'} \subseteq S_t$ for all $t \subseteq t'$ in T, and with the property that for every $t \in T$, if t is not a maximal node in T, then $\{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle\} \subseteq T$ and $\{S_{t^\frown\langle 0\rangle}, S_{t^\frown\langle 1\rangle}\}$ is a partition of S_t into stationary sets.

By $\subseteq$–maximality of $(S_t \,:\, t \in T)$ and the countable completeness of the nonstationary ideal it follows then that there is $X \subseteq T$ of size $\aleph_1$ definable from A such that $\{S_t \,:\, t \in X\}$ is a set of pairwise disjoint stationary sets, which contradicts our choice of $(S_i)_{i<\omega}$ and A. $\square$

It would be interesting to explore the possibilities for (other) large cardinal axioms to be equiconsistent with

"ZFC+$P(x)$ has definability strength strictly greater than $Q(x)$"

for other natural pairs of properties $P(x)$, $Q(x)$.

Recall that, for a nonzero $n \in \omega$, δ^1_n denotes the supremum of the lengths of all $\mathbf{\Delta}^1_n$–pre-wellorderings of the reals. It is not clear how to convert the proof of above theorem into a corresponding consistency result over ZF, but one can easily prove such results starting with a model of $\mathsf{ZF} + \mathsf{AD}$.

For example, a classical well–known result of Solovay is that, under AD, the club filter on $\delta^1_1 = \omega_1$ is an ultrafilter and therefore ω_1 cannot be partitioned into 2 stationary sets. The following theorem generalises this.

Theorem

($\mathsf{ZF} + \mathsf{AD}$) For every $n < \omega$, δ^1_{2n+1} is a successor cardinal and regular and, letting κ be such that $\kappa^+ = \delta^1_{2n+1}$, $\mathrm{Coll}(\omega, \kappa)$ forces that there is a partition of $(\delta^1_{2n+1})^{\mathbf{V}} = \omega_1$ into $2^{n+1} - 1$ stationary sets but no partition of ω_1 into more than $2^{n+1} - 1$ stationary sets.