A descriptive view of infinite dimensional group representations

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Two representations $\varphi : G \to GL_n(\mathbb{C})$ and $\psi : G \to GL_m(\mathbb{C})$ are ***equivalent*** *if $n = m$ and there exists $A \in GL_n(\mathbb{C})$ such that*

$$\psi(g) = A\,\varphi(g)\,A^{-1} \quad \text{for all } g \in G.$$

Finite Dimensional Irreducible Representations

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Theorem (Frobenius & Burnside 1904)

If G is a finite group, then the number of irreducible representations of G (up to equivalence) is equal to the number of conjugacy classes.

Arbitary Finite Dimensional Representations

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Theorem (Maschke & Schur 1905)

If G is a finite group, then every representation of G is uniquely expressible as a direct sum of irreducible representations.

Definition

If $\varphi : G \to GL_n(\mathbb{C})$ and $\psi : G \to GL_m(\mathbb{C})$ are representations, then the direct sum $(\varphi \oplus \psi) : G \to GL_{n+m}(\mathbb{C})$ is defined by

$$g \mapsto \begin{pmatrix} \varphi(g) & \mathbf{0} \\ \mathbf{0} & \psi(g) \end{pmatrix}$$

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Finite Dimensional Unitary Representations

Observation

If $\varphi : G \to U_n(\mathbb{C})$ *is a unitary representation and* $W \leqslant \mathbb{C}^n$ *is* G*-invariant, then so is the orthogonal complement*

$$W^\perp = \{\, v \in \mathbb{C}^n \mid \langle\, v, w \,\rangle = 0 \text{ for all } w \in W \,\}.$$

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Theorem (Loewy & Moore 1898)

If G is a finite group, then every representation of G is equivalent to a unitary representation.

Theorem (Folklore)

If G is a finite group and $\varphi, \psi : G \to U_n(\mathbb{C})$ *are unitary representations, then* φ, ψ *are equivalent iff* φ, ψ *are unitarily equivalent.*

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Proof.

If $m \geq 6$ and $\varphi : \mathrm{Alt}(m) \to GL_n(\mathbb{C})$ is a nontrivial representation, then $n \geq m - 1$. □

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Question

How about infinite dimensional representations of infinite groups?

Infinite Dimensional Representations

The Separable Hilbert Spaces

If X is a countable set, then

$$\ell^2(X) = \{\, a : X \to \mathbb{C} : \sum |a(x)|^2 < \infty \,\},$$

equipped with the inner product

$$\langle\, a, b \,\rangle = \sum a(x)\overline{b(x)}.$$

and the corresponding norm

$$||\, a \,|| = \sqrt{\sum |a(x)|^2}.$$

Definition

If G is a countable group, then a unitary representation of G is a homomorphism $\varphi : G \to U(\mathcal{H})$, where:

- *$\mathcal{H}$ is a separable complex Hilbert space; and*
- *$U(\mathcal{H})$ is the corresponding group of unitary transformations.*

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Example

We can define a unitary representation on

$$\ell^2(G) = \{ a : G \to \mathbb{C} : \sum |a(x)|^2 < \infty \}$$

by letting

$$(g \cdot a)(x) = a(g^{-1}x).$$

Unitary Representations of Countable Groups

Definition

Two representations $\varphi : G \to U(\mathcal{H})$ *and* $\psi : G \to U(\mathcal{H})$ *are* *unitarily equivalent* *if there exists* $A \in U(\mathcal{H})$ *such that*

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The unitary representation $\varphi : G \to U(\mathcal{H})$ is irreducible if there are no nontrivial proper G-invariant closed subspaces $0 < W < \mathcal{H}$.

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- *Can we classify the irreducible unitary representations of G up to unitary equivalence?*
- *Can we classify arbitrary unitary representations of G via "suitable decompositions" into irreducible representations?*

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Theorem (Folklore)

The irreducible unitary representations of $\mathbb{Z}$ are

$$\varphi_z : \mathbb{Z} \to U_1(\mathbb{C}) = \mathbb{T} = \{ c \in \mathbb{C} : |c| = 1 \}$$

where $z \in \mathbb{T}$ and $\varphi_z(k) = z^k$.

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Observation

The unitary representation $\mathbb{Z} \curvearrowright \ell^2(\mathbb{Z})$ has no $\mathbb{Z}$-invariant 1-dimensional subspaces.

The Unitary Representations of $\mathbb{Z}$

- The "multiplicity-free" unitary representations of $\mathbb{Z}$ can be parameterized by the probability measures μ on $\mathbb{T}$ so that the following are equivalent:
  - (ii) the measures μ, ν have the same null sets;
  - (i) the corresponding unitary representations φ_μ, φ_ν are unitarily equivalent.

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Definition

A Polish space is a separable completely metrizable topological space. E.g. $\mathbb{R}$, $\mathbb{C}$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$,...

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A First Approximation

If E is an equivalence relation on the set X, then a parameterization of X/E is an explicit map $f : X \to Z$ to a Polish space Z such that

$$f(x) = f(y) \iff x\, E\, y.$$

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- *Equivalently, if $f^{-1}(A)$ is Borel for each Borel subset $A \subseteq \mathbb{R}$.*

What is a parameterization?

Definition (Mackey)

An equivalence relation E on a Polish space X is parameterizable or smooth if there exists a Borel map $f : X \to Z$ to a Polish space Z such that

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Theorem (Freedman 1966)

If X is an uncountable Polish space, then the measure equivalence relation on the space $M(X)$ of probability measures on X is not smooth.

A “simple” non-smooth equivalence relation

Definition

E_0 is the equivalence relation on $2^{\mathbb{N}}$ defined by:

$$x \; E_0 \; y \iff x_n = y_n \quad \textit{for all but finitely many } n.$$

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Notation

Let μ be the uniform product probability measure on $2^{\mathbb{N}}$.

Theorem (Kolmogorov Zero-One Law 1933)

If $f : 2^{\mathbb{N}} \to [\,0, 1\,]$ is a Borel map which is constant on E_0-classes, then f is constant on a μ-measure 1 subset.

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Remark

In particular, an equivalence relation E is smooth iff E is Borel reducible to the identity relation Δ_Z on some Polish space Z.

The Glimm-Effos Dichotomy 1961/65

Theorem (Harrington-Kechris-Louveau 1990)

If E is a Borel equivalence relation on the Polish space X, then exactly one of the following holds:

(i) *E is smooth; or*

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An equivalence relation E on a Polish space X is Borel if E is a Borel subset of $X \times X$.

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- Let $\approx_G$ be the unitary equivalence relation defined on $\mathrm{Irr}_\infty(G)$ by

$$\varphi \approx_G \psi \iff (\, \exists A \in U(\mathcal{H})\,)\;(\, \forall g \in G\,)\;\; A\varphi(g)A^{-1} = \psi(g).$$

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Question

Is $\approx_G$ a smooth equivalence relation?

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Question

Does this mean that we should abandon all hope of finding a "satisfactory classification" for the irreducible unitary representations of these groups?

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Theorem (Hjorth 1997)

If the countable group G is not abelian-by-finite, then the action of $U(\mathcal{H})$ on $\mathrm{Irr}_\infty(G)$ is turbulent.

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Remarks

- This is a much more serious obstruction to the existence of a "satisfactory classification" of the irreducible unitary representations of G.
- But hopefully this is not the end of the story ...

An aside: Classifying homeomorphisms of $[0, 1]$

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Problem

Classify the elements of $\mathrm{Hom}^+([0,1])$ *up to conjugacy by "discrete invariants".*

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Definition

A **bump** *of* $\varphi \in \mathrm{Hom}^{+}([0,1])$ *is a maximal open interval* $I \subset [0,1]$ *such that one of the following conditions hold:*

(a) $\varphi(x) > x$ *for all* $x \in I$.

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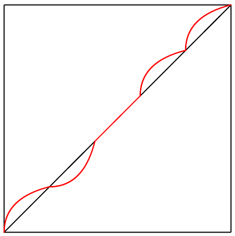

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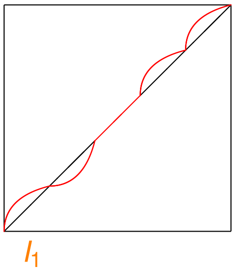

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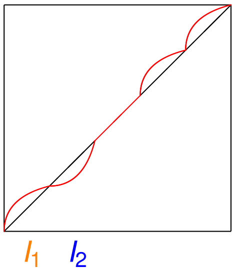

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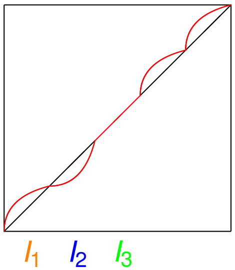

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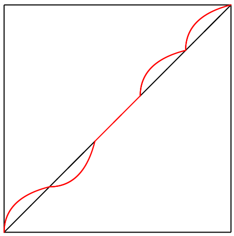

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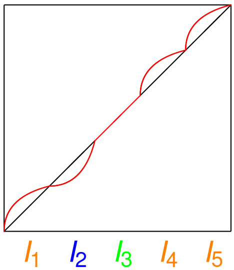

The Bump Structure

Definition

A **bump** *of* $\varphi \in \mathrm{Hom}^+([0,1])$ *is a maximal open interval* $I \subset [0,1]$ *such that one of the following conditions hold:*

(a) $\varphi(x) > x$ *for all* $x \in I$.

(b) $\varphi(x) < x$ *for all* $x \in I$.

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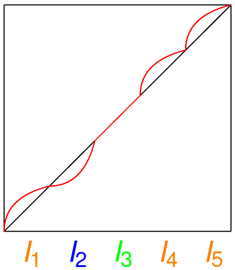

Theorem (Folklore)

Two maps $\varphi, \psi \in \mathrm{Hom}^+([0,1])$ *are conjugate iff the corresponding colored linear orders are isomorphic.*

Classifying homeomorphisms of $[0,1]^2$

Definition

$\mathrm{Hom}^+([0,1]^2)$ *is the group of homeomorphisms*

$$\varphi : [0,1]^2 \to [0,1]^2$$

satisfying $\varphi(v) = v$ *for each* $v \in \{(0,0),(0,1),(1,0),(1,1)\}$.

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The conjugacy relation on $\mathrm{Hom}^+([0,1]^2)$ *does not admit classification by "discrete invariants" ...*

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The conjugacy relation on $\mathrm{Hom}^+([0,1]^2)$ *does not admit classification by "discrete invariants" ... because the conjugacy relation is turbulent.*

When it's bad, it's worse ...

Theorem (Hjorth 1997)

If the countable group G is not abelian-by-finite , then the action of $U(\mathcal{H})$ *on* $\mathrm{Irr}_\infty(G)$ *is turbulent.*

Remarks

- This rules out the existence of a "satisfactory classification" of the irreducible unitary representations of G.
- But hopefully this is not the end of the story ...

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Open Question (Thomas 2011)

Do there exist countable groups G, H such that

(i) G, H are not abelian-by-finite; and

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Dixmier's Question

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Question (Dixmier 1967)

Do there exist countable groups G, H such that

(i) G, H are not abelian-by-finite; and

(ii) the unitary duals $\mathrm{Irr}_\infty(G)/\approx_G$ and $\mathrm{Irr}_\infty(H)/\approx_H$ are not Borel isomorphic?

Definition

If E, F are equivalence relations on the Polish spaces X, Y, then X/E, Y/F are Borel isomorphic if there exist Borel maps $\varphi : X \to Y$ and $\psi : Y \to X$ which induce mutually inverse bijections $\hat{\varphi}$, $\hat{\psi}$ between X/E and Y/F.

Borel bireducibility vs Borel isomorphism

Theorem (Motto Ros 2012)

If E, F are Borel orbit equivalence relations of actions of Polish groups on the Polish spaces X, Y, then the following are equivalent:

- *E and F are Borel bireducible.*
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Corollary

If G, H are countable groups, then the following are equivalent:

- *The unitary equivalence relations $\approx_G$ and $\approx_H$ are Borel bireducible.*
- *The unitary duals $\mathrm{Irr}_\infty(G)/\approx_G$ and $\mathrm{Irr}_\infty(H)/\approx_H$ are Borel isomorphic.*

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- *The free group* $\mathbb{F}_\infty$ *on infinitely many generators*
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- *The direct sum* Σ *of infinitely copies of* $\mathrm{Sym}(3)$

Representation universal groups

Definition

A countable group G is representation universal if $\approx_H$ is Borel reducible to $\approx_G$ for every countable group H.

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$\mathbb{F}_\infty$ is representation universal.

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Proof.

If $\theta : \mathbb{F}_\infty \to G$ is a surjective homomorphism, then the induced map

$$\mathrm{Irr}_\infty(G) \to \mathrm{Irr}_\infty(\mathbb{F}_\infty)$$
$$\varphi \mapsto \varphi \circ \theta$$

is a Borel reduction from $\approx_G$ to $\approx_{\mathbb{F}_\infty}$. □

Proposition

If G, H are countable groups and there exists a surjective homomorphism $G \to H$, then $\approx_H$ is Borel reducible to $\approx_G$.

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Does the existence of an injective homomorphism $G \hookrightarrow H$ imply that $\approx_G$ is Borel reducible to $\approx_H$?

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Open Problem

Is $SL(3, \mathbb{Z})$ representation universal?

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Making essential use of Elliott (1977) and Sutherland (1983)

Theorem (Thomas 2012)

If the countable group G is amenable and not abelian-by-finite, then G is representation minimal.

Amenable groups

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A countable group G is amenable if there exists a left-invariant finitely additive probability measure $\mu : \mathcal{P}(G) \to [\,0, 1\,]$.

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The "Obvious" Examples

The class $\mathcal{E}$ of elementary amenable groups is the smallest collection of countable groups such that:

- $\mathcal{E}$ contains all finite groups and all countable abelian groups.
- If $G \in \mathcal{E}$ and $H \leqslant G$, then $H \in \mathcal{E}$.
- If $G \in \mathcal{E}$ and $N \trianglelefteq G$, then $G/N \in \mathcal{E}$.
- If $N \trianglelefteq G$ and $N, G/N \in \mathcal{E}$, then $G \in \mathcal{E}$.
- $\mathcal{E}$ is closed under countable directed limits.

Some Non-Obvious Examples

- (Grigorchuk 1984) Groups of intermediate growth.

Amenable groups

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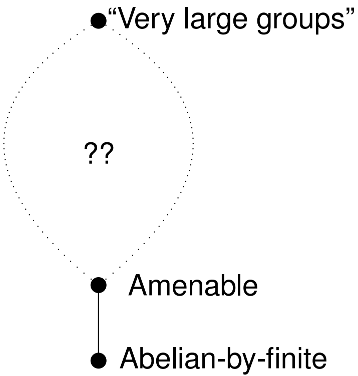
Remark

The proof ultimately depends upon the Ornstein-Weiss Theorem that if G, H are countable amenable groups, then any free ergodic measure-preserving actions of G, H are orbit equivalent.

Summing up



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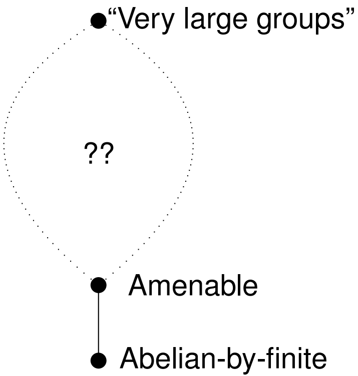

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$\mathbb{F}_2$ *is representation universal.*

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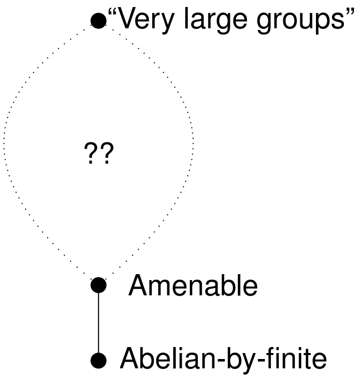

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A countable group G is very large if there exists a surjective homomorphism $G \to \mathbb{F}_2$.

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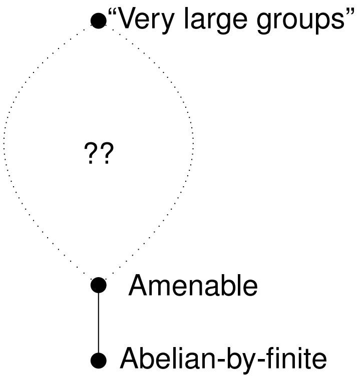

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A countable group G is very large if there exists a surjective homomorphism $G \to \mathbb{F}_2$.

Conjecture (Thomas 2016)

Every countable non-amenable group is representation universal.

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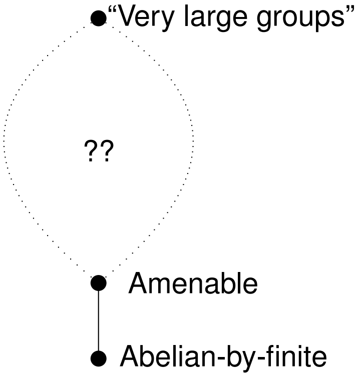

Theorem (Thomas 2012)

$\mathbb{F}_2$ is representation universal.

Definition

A countable group G is *very large* *if there exists a surjective homomorphism $G \to \mathbb{F}_2$.*

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Every countable non-amenable group is representation universal.

The End