The isomorphism and bi-embeddability relations for finitely generated groups

Simon Thomas

Rutgers University

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The starting point ...

Theorem (Thomas-Velickovic 1998)

The isomorphism relation $\cong$ on the space $\mathcal{G}_{fg}$ of finitely generated groups is countable universal.

Theorem (Williams 2012)

The bi-embeddability relation $\approx$ on the space $\mathcal{G}_{fg}$ of finitely generated groups is countable universal.

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Definition

A Borel equivalence relation E is countable if every E-class is countable.

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A countable Borel equivalence relation E is universal if $F \leq_B E$ for every countable Borel equivalence relation F.

The basic problem

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Determine the Borel complexity of the isomorphism and bi-embeddability relations for various restricted classes of finitely generated groups, such as:

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- simple groups
- Kazhdan groups

What is a simple group?

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A group G is **simple** if its only normal subgroups are 1 and G.

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Proof.

A group G is simple if and only if for every $1 \neq g \in G$, the conjugacy class $g^G = \{ xgx^{-1} \mid x \in G \}$ generates G. $\square$

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Corollary (Shalom 2000)

Every finitely generated Kazhdan group is a homomorphic image of a finitely presented Kazhdan group.

The Polish space of finitely generated groups

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- For each $m \geq 1$, let $\mathcal{G}_m$ be the set of isomorphism types of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.
- Then there exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1_G)).$$

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- And $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ is the space of f.g. groups.

The Polish space of finitely generated groups

- Let $(G, \bar{s}) \in \mathcal{G}_m$ and let d_S be the corresponding word metric. For each $\ell \geq 1$, let

$$B_\ell(G, \bar{s}) = \{g \in G \mid d_S(g, 1_G) \leq \ell\}.$$

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- The basic open neighborhoods of $(G, \bar{s})$ in $\mathcal{G}_m$ are given by

$$U_{(G,\bar{s}),\ell} = \{\, (H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s}) \,\}, \qquad \ell \geq 1.$$

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Observation

The isomorphism and bi-embeddability relations on the space $\mathcal{G}_{fg}$ of finitely generated groups are both countable Borel.

The Polish space of finitely generated groups

Proposition

For each $G \in \mathcal{G}_{fg}$, there exists a sequence of finitely presented groups $G_n \in \mathcal{G}_{fg}$ such that:

- $G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \cdots \twoheadrightarrow G_n \twoheadrightarrow \cdots \twoheadrightarrow G$.
- *G is a homomorphic image of each G_n.*
- $\lim_{n \to \infty} G_n = G$.

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Proof.

- Let $G = (G, \bar{s}) \in \mathcal{G}_{fg}$.

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Proof.

- Let $G = (G, \bar{s}) \in \mathcal{G}_{fg}$.
- Let R_n be the identities $w(\bar{s}) = 1$ which are visible in $B_n(G, \bar{s})$.

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Proof.

- Let $G = (G, \bar{s}) \in \mathcal{G}_{fg}$.
- Let R_n be the identities $w(\bar{s}) = 1$ which are visible in $B_n(G, \bar{s})$.
- Then $G_n = \langle\, \bar{s} \mid R_n \,\rangle$ satisfies our requirements.

□

The Polish space of finitely generated groups

Corollary

If $\mathcal{K}$ is an open subset of $\mathcal{G}_{fg}$, then every $G \in \mathcal{K}$ is a homomorphic image of a finitely presented group $H \in \mathcal{K}$.

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Proof.

Let (G_n) be a sequence of finitely presented groups such that:

- G is a homomorphic image of G_n.
- $\lim_{n \to \infty} G_n = G$.

Then $G_n \in \mathcal{K}$ for all but finitely many $n \in \mathbb{N}$. □

The Main Theorems

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Definition

The Borel equivalence relation E is smooth if E is Borel reducible to the identity relation Δ_X on some (equivalently every) uncountable Polish space X.

Weakly universal Borel equivalence relations

Definition

A countable Borel equivalence relation E is weakly universal if for every countable Borel equivalence relation F, there exists a countable-to-one Borel homomorphism from F to E.

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Theorem (Kechris-Miller 2008)

A countable Borel equivalence relation E is weakly universal iff there exists a universal countable Borel equivalence relation $F \subseteq E$.

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A countable Borel equivalence relation E is weakly universal if for every countable Borel equivalence relation F, there exists a countable-to-one Borel homomorphism from F to E.

Example

The Turing equivalence relation $\equiv_T$ on $2^{\mathbb{N}}$ is weakly universal.

Weakly universal Borel equivalence relations

Question

Does there exist a weakly universal Borel equivalence relation which is not countable universal?

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Theorem (Dougherty & Kechris 1999)

If Martin's Conjecture on degree invariant Borel maps holds, then the Turing equivalence relation $\equiv_T$ is weakly universal but not countable universal.

The main theorems

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Initial Target

The bi-embeddability relation $\approx$ on the space $\mathcal{G}_{fg}$ of finitely generated groups is weakly universal.

The word problem for finitely generated groups

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For each $m \geq 1$, fix a recursive enumeration $\{ w_k(x_1, \cdots, x_m) \mid k \in \mathbb{N} \}$ of the (not necessarily reduced) words in $x_1, \cdots, x_m, x_1^{-1}, \cdots, x_m^{-1}$.

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Definition

If $(G, \bar{s}) \in \mathcal{G}_m$ is a finitely generated group, then

$$\mathrm{Rel}(G) = \{ k \in \mathbb{N} \mid w_k(s_1, \cdots, s_m) = 1 \}$$
$$\mathrm{Nonrel}(G) = \{ k \in \mathbb{N} \mid w_k(s_1, \cdots, s_m) \neq 1 \}.$$

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Question

But what if we choose a different generating set $G = \langle t_1, \cdots, t_n \rangle$?

The most obvious Turing reductions

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is one-one reducible to B, written $A \leq_1 B$, if there exists an injective recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$n \in A \iff f(n) \in B.$$

Definition

If $A \leq_1 B$ and $B \leq_1 A$, then we write $A \equiv_1 B$ and say that A, B are recursively isomorphic.

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Example

If $(G, \bar{s}), (H, \bar{t}) \in \mathcal{G}_{fg}$ and $G \hookrightarrow H$, then $\mathrm{Rel}(G, \bar{s}) \leq_1 \mathrm{Rel}(H, \bar{t})$ and $\mathrm{Nonrel}(G, \bar{s}) \leq_1 \mathrm{Nonrel}(H, \bar{t})$.

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Definition

If $A \in 2^{\mathbb{N}}$, then the finitely generated group G is relatively universal of degree A if $\mathrm{Sk}(G) = \{\, H \in \mathcal{G}_{fg} \mid \mathrm{Rel}(H) \leq_T A \,\}$.

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If $A \in 2^{\mathbb{N}}$, then the finitely generated group G is relatively universal of degree A if $\mathrm{Sk}(G) = \{\, H \in \mathcal{G}_{fg} \mid \mathrm{Rel}(H) \leq_T A \,\}$.

Observation

If $A, B \in 2^{\mathbb{N}}$ and G_A, G_B are relatively universal of degrees A, B, then

$$G_A \approx G_B \iff A \equiv_T B.$$

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Theorem

For each f.g. group G, there exists a f.g. group H such that:

- $\mathrm{Rel}(H) \leq_T \mathrm{Rel}(G)$; *and*
- *H does not embed into G.*

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- Let $\{ \langle u_k(x_1, \cdots, x_n), v_k(x_1, \cdots, x_n) \rangle \mid k \in \mathbb{N} \}$ be a recursive enumeration of the ordered pairs of words in $x_1, \cdots, x_n$, $x_1^{-1}, \cdots, x_n^{-1}$.

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- For each $k \in \mathbb{N}$, let $r_k(x, y)$ be the word $(\, x^{k+1} y^{k+1} \,)^7$.

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- For each $k \in \mathbb{N}$, let $r_k(x, y)$ be the word $(x^{k+1} y^{k+1})^7$.
- Let H have presentation $\langle b, c \mid R \rangle$, where

$$r_k(b, c) \in R \iff r_k(u_k(a_1, \cdots, a_n), v_k(a_1, \cdots, a_n)) \neq 1$$

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- If $\varphi : H \hookrightarrow G$, then there exists $k \in \mathbb{N}$ such that $\varphi(b) = u_k(a_1, \cdots, a_n)$ and $\varphi(c) = v_k(a_1, \cdots, a_n)$.

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- Clearly $r_k(b, c) \notin R$ and so $r_k(u_k(a_1, \cdots, a_n), v_k(a_1, \cdots, a_n)) = 1$.
- But since $r_k(b, c) \notin R$, it follows that $r_k(b, c) \neq 1$ in H, which is a contradiction.

Enumeration reducibility

Definition

If $A, B \subseteq \mathbb{N}$, then A is enumeration reducible to B, written $A \leq_e B$, if there exists a recursively enumerable subset $R \subseteq \mathbb{N} \times \mathrm{Fin}(\mathbb{N})$ such that

$$n \in A \iff (n, F) \in R \text{ for some finite subset } F \subseteq B.$$

Remark

Intuitively, $A \leq_e B$ if there is a fixed effective procedure which produces an enumeration of A from any enumeration of B.

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Observation

It is easily checked that if $A \leq_1 B$, then $A \leq_e B$.

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Remark

If $G, H \in \mathcal{G}_{fg}$ and $G \hookrightarrow H$, then $\mathrm{Rel}(G) \leq_e \mathrm{Rel}(H)$ and $\mathrm{Nonrel}(G) \leq_e \mathrm{Nonrel}(H)$.

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Theorem (Folklore)

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Proof.

$$A \leq_T B \iff A \oplus (\mathbb{N} \smallsetminus A) \leq_e B \oplus (\mathbb{N} \smallsetminus B).$$

□

Relatively universal f.g. groups revisited

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If $A, B \in 2^{\mathbb{N}}$ and G_A, G_B are relatively universal of e-degrees A, B, then

$$G_A \approx G_B \quad \Longleftrightarrow \quad A \equiv_e B.$$

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Observation

If $A, B \in 2^{\mathbb{N}}$ and G_A, G_B are relatively universal of e-degrees A, B, then

$$G_A \approx G_B \iff A \equiv_e B.$$

Theorem (Higman-Scott 1998)

For each $A \in 2^{\mathbb{N}}$, there exists a relatively universal group G_A of e-degree A.

Two contrasting results

Theorem

There exists a Borel map $A \mapsto G_A$ from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that:

- $\mathrm{Rel}(G_A) \equiv_e A$, *and*
- *if* $A \equiv_e B$, *then* $G_A \approx G_B$.

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Theorem (Thomas 2010)

If $A \mapsto G_A$ *is a Borel map from* $2^{\mathbb{N}}$ *to* $\mathcal{G}_{fg}$ *such that* $\mathrm{Rel}(G_A) \equiv_T A$, *then there exists a Turing degree* $\mathbf{d}_0$ *such that for all Turing degrees* $\mathbf{d} \geq_T \mathbf{d}_0$, *there exists an infinite subset* $\{ A_n \mid n \in \mathbb{N} \} \subseteq \mathbf{d}$ *such that the groups* $\{ G_{A_n} \mid n \in \mathbb{N} \}$ *are pairwise incomparable with respect to embeddability.*

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Theorem (Thomas-Williams 2013)

For each $A \in 2^{\mathbb{N}}$, there exists a finitely generated group K_A such that

- *K_A is relatively universal of e-degree A; and*
- *K_A is a Kazhdan group.*

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Corollary

The bi-embeddability relation for Kazhdan groups is weakly universal.

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- By Ol'shanskii, H has a free subgroup $F = \langle a_1, \cdots, a_n \rangle$ such that for every $N \trianglelefteq F$, there exists $M \trianglelefteq H$ such that $N = M \cap F$.

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- Let $a_i = u_i(\bar{x})$ for $1 \leq i \leq n$.
- Let $K_A = \langle x_1, \cdots, x_m \mid R \rangle$, where

$$R = T \cup \{ w(u_1(\bar{x}), \cdots, u_n(\bar{x})) \mid w(a_1, \cdots, a_n) \in \mathrm{Rel}(G_A) \}.$$

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Definition

A finitely generated group $G \in \mathcal{G}_{fg}$ is recursively presented if $\mathrm{Rel}(G)$ is recursively enumerable.

Theorem (Kuznetsov 1958)

If G is a recursively presented simple group, then G has a solvable word problem.

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Theorem (Thomas-Williams 2013)

If $A \in 2^{\mathbb{N}}$ and G_A is relatively universal of e-degree A, then G_A is not simple.

Proof of the Extended Kuznetsov Theorem

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- Let $G = \langle a_1, \cdots, a_n \rangle$ be a finitely generated simple group.
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- Fix some word $u \in W$ such that $u(a_1, \cdots, a_n) \neq 1$ in G.
- Since G is simple, if $w \in W$, then the following are equivalent:
  - (i) $w \in \text{Nonrel}(G)$.
  - (ii) The group G_w with presentation $\langle x_1, \cdots, x_n \mid \text{Rel}\, G \cup \{ w \} \rangle$ is trivial.
  - (iii) u is in the normal closure of $\text{Rel}(G) \cup \{ w \}$ in $\mathbb{F}_n$.

Proof of the Extended Kuznetsov Theorem

- Consider the set $R \subseteq W \times \text{Fin}(W)$ defined by

$$(w, F) \in R \iff u \text{ is in the normal closure of } F \cup \{ w \} \text{ in } \mathbb{F}_n.$$

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- Then R is recursively enumerable and the following are equivalent:
  (i) $w \in \text{Nonrel}(G)$.
  (ii) $(w, F) \in R$ for some finite subset $F \subseteq \text{Rel}(G)$.

- Thus $\text{Nonrel}(G) \leq_e \text{Rel}(G)$.

A characterization of the total enumeration degrees

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Definition

A set $A \in 2^{\mathbb{N}}$ is total if $A \equiv_e A \oplus (\mathbb{N} \smallsetminus A)$.

Example

If $G \in \mathcal{G}_{fg}$ is simple, then $\mathrm{Rel}(G)$ is total.

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An enumeration degree $\mathbf{e}$ is total if it contains a total set.

Theorem (Thomas 2016)

An enumeration degree $\mathbf{e}$ *is total if and only if there exists a finitely generated simple group* G *such that* $\text{Rel}(G) \in \mathbf{e}$.

The obvious conjectures ...

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The isomorphism relation for finitely generated simple groups is not weakly universal.

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There does not exist an isomorphism-invariant Borel map $\phi : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ *such that for all* $G \in \mathcal{G}_{fg}$,

- $\phi(G)$ *is a finitely generated simple group; and*
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There does not exist an isomorphism-invariant Borel map $\phi : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ *such that for all* $G \in \mathcal{G}_{fg}$,

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The obvious conjectures ...

Conjecture

The isomorphism relation for finitely generated simple groups is not weakly universal.

Theorem

There exists an isomorphism-invariant Borel map $\phi : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ *such that for all* $G \in \mathcal{G}_{fg}$,

- $\phi(G)$ *is a 2-generator group; and*
- $G \hookrightarrow \phi(G)$.

On the other hand ...

Notation

$\mathcal{G}$ is the space of arbitrary countable groups.

Theorem (Ramsey cardinal)

- *Suppose that $G \mapsto K_G$ is any Borel map from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$.*
- *Then there exists an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to relative constructibility; i.e., if $G \neq H \in \mathcal{F}$, then $K_G \notin L[K_H]$ and $K_H \notin L[K_G]$.*

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Remark

In *ZFC*, we can find an uncountable Borel family $\mathcal{F}$ such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to embeddability.

Definition

An infinite group G is said to be just infinite if every proper quotient of G is finite.

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Remark

An interesting theory of just infinite groups has been developed by Girgorchuk, Wilson, etc.

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Proposition (Grigorchuk 2000)

Every infinite f.g. group G has a just infinite quotient G/N.

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- It is enough to show that the partially ordered set

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has a maximal element.

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- Suppose that $N_0 \leqslant \cdots \leqslant N_\ell \leqslant \cdots$ is a chain and let $N = \bigcup N_\ell$.
- If $N \notin \mathcal{N}$, then $[\, G : N\,] < \infty$ and this implies that N is f.g., which is a contradiction.

□

Just infinite groups

Theorem (Thomas 2013)

There does not exist a Borel map $G \mapsto Q_G$ from $\mathcal{G}_{fg}$ to $\mathcal{G}_{fg}$ such that for all $G, H \in \mathcal{G}_{fg}$,

- *Q_G is a just infinite quotient of G; and*
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This is an easy consequence of:

Theorem (Thomas 2013)

The isomorphism relation on the space of finitely generated simple groups is not smooth.

A nonselection result

Notation

If $x \in 2^{\mathbb{N}}$, *then* $\bar{x}(n) = 1 - x(n)$.

Definition

E_0^* is the Borel equivalence relation on $2^{\mathbb{N}}$ defined by

$$x \; E_0^* \; y \iff x \; E_0 \; y \text{ or } x \; E_0 \; \bar{y}.$$

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Proposition (Folklore)

There does not exist a Borel map $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *such that:*

- *if* $x \; E_0^* \; y$*, then* $\theta(x) \; E_0 \; \theta(y)$*;*
- $\theta(x) \; E_0^* \; x$.

Proof of Proposition

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$$X = \{ x \in 2^{\mathbb{N}} \mid x \; E_0 \; y \text{ for some } y \in \theta[2^{\mathbb{N}}] \}.$$

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- Then X is a Borel tail event and hence $\mu(X) = 0, 1$.
- Since the map $x \mapsto \bar{x}$ is measure preserving, it follows that $\mu(2^{\mathbb{N}} \smallsetminus X) = \mu(X)$, which is impossible.

Proof of Theorem

- Suppose $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ is a Borel map such that
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- Thus the Borel map $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by

$$\theta(z) = y \iff y \in \{ z, \bar{z} \} \text{ and } \varphi(G_z) \cong S_y$$

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The End