A Direct Proof of Schwichtenberg's Bar Recursion Closure Theorem

Silvia Steila

on-going work with Paulo Oliva and Stefano Berardi

Universität Bern

May 27th, 2016

System T

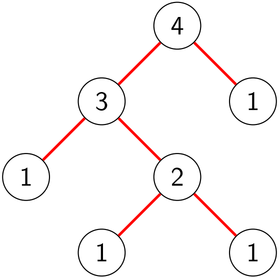
- System T consists of the simply typed λ-calculus, enriched with
  - natural numbers (0 and Succ);
  - primitive recursion Rec in all finite types;
  - together with the associated reduction rules.

- The finite types are defined inductively:
  - $\mathbb{N}$ is the basic finite type;
  - $\tau_0 \to \tau_1$ is the type of functions from τ_0 to τ_1;
  - τ_0^* is the type of finite sequences whose elements are of type τ_0.

- For any finite type, its type level is:
  - $\mathrm{tpl}(\mathbb{N})=0$;
  - $\mathrm{tpl}(\rho \to \eta)=\max(\mathrm{tpl}(\rho)+1, \mathrm{tpl}(\eta))$;
  - $\mathrm{tpl}(\tau^*)= \mathrm{tpl}(\tau)$.

Spector's bar recursion

Spector's bar recursion can be explained as a recursive definition of a function through the set of the nodes of a well-founded tree.



Given a well-founded tree we first define the value of the function on the leaves and then we calculate the value on a node by using the values of its immediate children.

Spector's bar recursion

Spector's bar recursion can be explained as a recursive definition of a function through the set of the nodes of a well-founded tree.

For any τ, σ, $G : \tau^* \to \sigma$, $H : \tau^* \to (\tau \to \sigma) \to \sigma$ and $Y : (\mathbb{N} \to \tau) \to \mathbb{N}$, Spector added to system T constants for bar recursion:

$$\mathrm{BR}^{\tau,\sigma}(G, H, Y)(s) \overset{\sigma}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(\lambda x^\tau.\mathrm{BR}(G, H, Y)(s * x)) & \text{otherwise.} \end{cases}$$

If Y is continuous, i.e.

$$\forall\alpha\exists n\forall\beta((\forall m < n(\alpha(m) = \beta(m))) \implies Y(\alpha) = Y(\beta)),$$

then the tree $\{s : Y(\hat{t}) \geq |t| \text{ for all t prefix of s}\}$ is well-founded.

Spector's bar recursion

Spector's bar recursion can be explained as a recursive definition of a function through the set of the nodes of a well-founded tree.

For any τ, σ, $G : \tau^* \to \sigma$, $H : \tau^* \to (\tau \to \sigma) \to \sigma$ and $Y : (\mathbb{N} \to \tau) \to \mathbb{N}$, Spector added to system T constants for bar recursion:

$$\mathrm{BR}^{\tau,\sigma}(G, H, Y)(s) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(\lambda x^{\tau}.\mathrm{BR}(G, H, Y)(s * x)) & \text{otherwise.} \end{cases}$$

If Y is continuous, i.e.

$$\forall \alpha \exists n \forall \beta ((\forall m < n(\alpha(m) = \beta(m))) \implies Y(\alpha) = Y(\beta)),$$

then the tree $\{s : Y(\hat{t}) \geq |t| \text{ for all t prefix of s}\}$ is well-founded.

Schwichtenberg's result

Theorem (Schwichtenberg, 1979)

System T is closed under bar recursion of type levels 0 and 1.

I.e. by using bar recursion of such types we can only define functionals which are already in system T.

Schwichtenberg's original proof is based on the notion of infinite terms as introduced by Tait.

Bar recursions of type levels 0 and 1 are reducible to α-recursion for some $\alpha < \varepsilon_0$. Hence, using an interdefinability result from Tait, they are also reducible to primitive recursions of higher types.

A witness for Schwichtenberg's result

The last year we provided a bar recursive bound for the Termination Theorem by Podelski and Rybalchenko.

The Termination Theorem characterizes the termination of transition-based program as a property of well-founded relations.

By using closure of BR for type 0, we proved that under certain hypotheses our bound is in system T.

We would obtain an explicit construction of the bound in system T.

If Y is constant

Proposition

$\lambda G, H, s.\mathrm{BR}^{\tau,\sigma}(G, H, \lambda\alpha.k)(s)$ is in T.

First define $\varphi(G, H)(n) \colon \tau^* \to \sigma$ by primitive recursion as

$$\varphi(G, H)(n) = \lambda s. \begin{cases} G(s) & \text{if } n = 0 \\ H(s)(\lambda x.\varphi(G, H)(n-1)(s * x)) & \text{otherwise.} \end{cases}$$

Then, define Ψ by cases, using φ, as

$$\Psi(G, H, k)(s) = \begin{cases} G(s) & \text{if } k < |s| \\ \varphi(G, H)(k + 1 - |s|)(s) & \text{otherwise.} \end{cases}$$

By bar induction we can prove that $\mathrm{BR}^{\tau,\sigma}(G, H, \lambda\alpha.k) = \Psi(G, H, k)$.

Secure bar recursion (sBR)

For each τ and for each $Y : (\mathbb{N} \to \tau) \to \mathbb{N}$, define the secure bar recursion

$$\mathsf{sBR}^{\tau,\sigma}(G, H, Y)(s) \stackrel{\sigma}{=}$$

$$\begin{cases} G(s) & \text{if } \lambda\beta.Y(s * \beta) \text{ is constant} \\ H(s)(\lambda x^{\tau}.\mathsf{sBR}^{\tau,\sigma}(G, H, Y)(s * x)) & \text{otherwise} \end{cases}$$

How to T-define BR from sBR

Proposition

$\mathrm{BR}^{\tau,\sigma}(Y)$ is T-definable in $\mathrm{sBR}^{\tau,\sigma}(Y)$.

Let Y be given. Define

$$\Phi(G,H,s) \stackrel{\sigma}{=} \begin{cases} \mathrm{BR}^{\tau,\sigma}(G,H,\lambda\beta.Y(\hat{s}))(s) & \text{if } \lambda\beta.Y(s*\beta) \text{ is constant} \\ H'(G,H,s)(\lambda x^{\tau}.\Phi(G,H,s*x)) & \text{otherwise} \end{cases}$$

where

$$H'(G,H,s)(f^{\tau\to\sigma}) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(f) & \text{otherwise.} \end{cases}$$

We prove by bar induction and continuity that for all s

$$P(s) \quad \equiv \quad \Phi(G,H,s) = \mathrm{BR}^{\tau,\sigma}(G,H,Y)(s)$$

using the bar condition $B(s) \quad \equiv \quad Y(\hat{s}) < |s|$.

When is sBR T-definable?

The last step is to prove that for $\tau = \mathbb{N}$ or $\tau = \mathbb{N} \to \mathbb{N}$ and for any fixed term $t[\alpha]$, $\mathsf{sBR}^{\tau,\sigma}(\lambda\alpha.t)$ itself is T-definable.

Intuitively, given a term $t[\alpha] : \mathbb{N}$ we will define a closed term $t^\circ : \mathbb{N}^\circ$ so as to have

$$t^\circ = \langle \lambda\alpha.t, \mathsf{sBR}_t^{\tau,\sigma} \rangle.$$

For terms t of higher types we define t° so that this property is preserved at ground type.

When is sBR T-definable?

Let $\Psi(G, H, k)$ be the primitive recursive term which defines $\mathrm{BR}(G, H, \lambda\alpha.k)$. For any term $t \colon \rho$ in system T, define the $t^\circ \colon \rho^\circ$ inductively as follows:

$$\begin{aligned}
0^\circ &= \langle \lambda\alpha.0, \lambda G, H, s.G(s)\rangle \\
\mathrm{Succ}^\circ &= \lambda\langle\phi, \Phi\rangle.\langle\lambda\alpha.\mathrm{Succ}(\phi(\alpha)), \Phi\rangle \\
\alpha^\circ &= \lambda\langle\phi, \Phi\rangle.\langle\lambda\alpha.\alpha(\phi(\alpha)), \lambda G, H, s.\Phi(\Psi(G, H, \phi(\hat{s})), H, s)\rangle \\
(\lambda x.t)^\circ &= \lambda x^\circ.t^\circ \\
(uv)^\circ &= u^\circ v^\circ \\
(\mathrm{Rec}^\rho)^\circ &= \lambda\langle\phi, \Phi\rangle, a^{\rho^\circ}, F^{\mathbb{N}^\circ \to \rho^\circ \to \rho^\circ}, v^{\eta^\circ}.\langle\lambda\alpha.\pi_0(r[\alpha])(\alpha), \Phi'\rangle.
\end{aligned}$$

where in the case of the Rec we assume $\rho = \eta \to \mathbb{N}$, and

- $r[\alpha] = \mathrm{Rec}(\phi(\alpha), a, \lambda k^{\mathbb{N}}, b^{\rho^\circ}.F(\langle\lambda\beta.k, \lambda G', H', s'.G'(s')\rangle, b))(v)$
- $\Phi' = \lambda G, H, s.\Phi(\pi_1(r[\hat{s}])(G, H), H, s)$.

When is sBR $\mathbb{T}$-definable?

Theorem

Let τ be of type 0 or 1. Let $t : \mathbb{N}$ with only one free variable α, then

$$\mathrm{sBR}^{\tau,\sigma}(G, H, \lambda\alpha.t)(s) = \pi_1(t^{\circ})(G, H, s).$$

Why has τ to be either of type 0 or of type 1?

Modulo of continuity

Given $Y\colon (\mathbb{N} \to \tau) \to \mathbb{N}$ a modulus of continuity for Y is a functional $\omega_Y\colon (\mathbb{N} \to \tau) \to \mathbb{N}$ such that

$$\forall\alpha\forall\beta((\forall m < \omega_Y(\alpha)(\alpha(m) = \beta(m))) \implies Y(\alpha) = Y(\beta)).$$

If τ has type level 0 or 1, then any T-definable $Y\colon (\mathbb{N} \to \tau) \to \mathbb{N}$ has a T-definable modulus of continuity.

There are T-definable terms for τ of level 2 for which does not exists a continuous modulus of continuity.

Why has τ to be either of type 0 or of type 1?

Theorem

Let τ be of type 0 or 1. Let $t : \mathbb{N}$ with only one free variable α, then

$$\mathsf{sBR}^{\tau,\sigma}(G, H, \lambda\alpha.t)(s) = \pi_1(t^\circ)(G, H, s).$$

The proof is by induction on the structure of t. In the step α we use bar induction for

$$B(s) \quad \equiv \quad \lambda\beta.t[s * \beta/\alpha] \text{ is constant}$$

Why has τ to be either of type 0 or of type 1?

$B(s) \equiv \lambda\beta.t[s * \beta/\alpha]$ is constant

There exists a modulus of continuity for $\lambda\alpha.t$

Let $B, \ P$ be subsets of τ^* such that

1. B is decidable;
2. B is downward closed for extension;
3. B contains a finite prefix of every infinite sequence;
4. P includes B;
5. P is inductive, i.e. $\forall s \in \tau^*[(\forall x \in \tau P(s * \langle x \rangle)) \implies P(s)]$.

Then $P(\langle\rangle)$.

Question

Can we define t° for any τ in such a way we have

$$\mathrm{sBR}^{\tau,\sigma}(G, H, \lambda\alpha.t)(s) = \pi_1(t^\circ)(G, H, s)?$$

Question

Can we define t° for any τ in such a way we have

$$\mathsf{sBR}^{\tau,\sigma}(G, H, \lambda\alpha.t)(s) = \pi_1(t^\circ)(G, H, s)?$$

Vielen Dank!