From equivalent forms of CH to CH-systems

Silvia Steila
on-going work with Raphaël Carroy and Alessandro Andretta

Universität Bern

Mini-Workshop

Bern
March 24, 2015

Törnquist and Weiss idea

In 2012 Törnquist and Weiss studied many Σ^1_2 definable version of some statements equivalent to CH ($2^{\aleph_0} = \aleph_1$).

CH $\iff$ there exist some **objects** such that **something happens**.

They proved that these Σ^1_2 counterparts become equivalent to the statement "all reals are constructible".

$\mathbb{R} \subseteq L$ $\iff$ there exist some Σ^1_2 **objects** such that **something happens**.

From CH-equivalences to $(\mathbb{R} \subseteq L)$-equivalences

Theorem (Sierpinski 1965)

CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there are Σ^1_2 sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

From CH-equivalences to $(\mathbb{R} \subseteq L)$-equivalences

Theorem (Sierpinski 1965)

CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ holds iff there are Σ^1_2 sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

From CH-equivalences to $(\mathbb{R} \subseteq L)$-equivalences

Theorem (Komjáth and Totik 2006)

CH holds iff there exists $g : \mathbb{R} \times \mathbb{R} \to \omega$ such that there are no two sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \restriction C \times D$ is monochromatic.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there exists a Σ^1_2-definable function $g : \mathbb{R} \times \mathbb{R} \to \omega$ such that there are no sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \restriction C \times D$ is monochromatic.

From CH-equivalences to $(\mathbb{R} \subseteq L)$-equivalences

Theorem (Komjáth and Totik 2006)

CH holds iff there exists a coloring $g : \mathbb{R} \to \omega$ such that there are no four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there exists a Σ^1_2 coloring $g : \mathbb{R} \to \omega$ such that there are no four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

From CH-equivalences to $(\mathbb{R} \subseteq L)$-equivalences

Theorem (Erdős and Kakutani 1943)

CH holds iff the set of all real numbers can be decomposed into a countable number of rationally independent subsets.

A set $X \subseteq \mathbb{R}$ is rationally independent if either $X = \{0\}$ or for any $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$ and for any $q_0, \ldots, q_{n-1} \in \mathbb{Q} \setminus \{0\}$ we have

$$\sum_{i=0}^{n-1} q_i x_i \neq 0.$$

From CH-equivalences to $(\mathbb{R} \subseteq L)$-equivalences

Theorem (Erdős and Kakutani 1943)

CH holds iff the set of all real numbers can be decomposed into a countable number of rationally independent subsets.

Theorem

$\mathbb{R} \subseteq L$ iff the set of all real numbers can be decomposed into a countable number of uniformly Σ^1_2 definable rationally independent subsets.

From CH-equivalences to ($\mathbb{R} \subseteq L$)-equivalences

Theorem (Erdős and Komjáth 1990)

CH holds if and only if the plane can be colored with countably many colors with no monochromatic right-angled triangle.

Theorem

$\mathbb{R} \subseteq L$ if and only if there exists a Σ^1_2 coloring of the plane with countably many colors with no monochromatic right-angled triangle.

CH-systems

P is a CH-system if

- $P \subseteq \omega \times \omega_1 \times \mathbb{R}$;
- $\forall i \exists m_i \forall S \in [\omega_1]^{m_i} (\bigcap \{P_{i\alpha} \mid \alpha \in S\}$ is countable);
- $\forall a (\{\alpha \in \omega_1 \mid \forall i \neg P(i, \alpha, a)\}$ is countable).

Picture on the blackboard.

CH-systems

P is a CH-system if

- $P \subseteq \omega \times \omega_1 \times \mathbb{R}$;
- $\forall i \exists m_i \forall S \in [\omega_1]^{m_i} (\bigcap \{P_{i\alpha} \mid \alpha \in S\}$ is countable);
- $\forall a (\{\alpha \in \omega_1 \mid \forall i \neg P(i, \alpha, a)\}$ is countable).

Theorem

If there exists a CH-system then CH holds.

CH-systems

P is a CH-system if

- $P \subseteq \omega \times \omega_1 \times \mathbb{R}$;
- $\forall i \exists m_i \forall S \in [\omega_1]^{m_i} (\bigcap \{P_{i\alpha} \mid \alpha \in S\}$ is countable);
- $\forall a (\{\alpha \in \omega_1 \mid \forall i \neg P(i, \alpha, a)\}$ is countable).

Fact

CH implies that there exists a CH-system.

Take an injection $f : \omega_1 \to \mathbb{R}$. For any i put

- $m_i = 1$,
- $P(i, \alpha, y) \equiv f(\alpha) \succ y$.

From CH-equivalences to CH-systems

Theorem (Sierpinski 1965)

CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Take an injection $f : \omega_1 \to \mathbb{R}$. For any i put

- $m_i = 1$,
- $P(i, \alpha, y) \equiv (f(\alpha), y) \in A$.

From CH-equivalences to CH-systems

Theorem (Sierpinski 1965)

CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

Take an injection $f : \omega_1 \to \mathbb{R}$. For any i put

- $m_i = 1$,
- $P(i, \alpha, y) \equiv \exists n((f(\alpha), y, n) \in A_2)$.

From CH-equivalences to CH-systems

Theorem (Komjáth and Totik 2006)

CH holds iff there exists $g : \mathbb{R} \times \mathbb{R} \to \omega$ such that there are no two sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \restriction C \times D$ is monochromatic.

Take an injection $f : \omega_1 \to \mathbb{R}$. For any i put

- $m_i = 2$,
- $P(i, \alpha, y) \equiv g(f(\alpha), y) = i$.

From CH-equivalences to CH-systems

Theorem (Komjáth and Totik 2006)

CH holds iff there exists a coloring $g : \mathbb{R} \to \omega$ such that there are no four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

Take an injection $f : \omega_1 \to \mathbb{R}$. For any i put

- $m_i = 2$,
- $P(i, \alpha, y) \equiv g(f(\alpha) + y) = i$.

From CH-equivalences to CH-systems

Theorem (Erdős and Kakutani 1943)

CH holds iff the set of all real numbers can be decomposed into a countable number of rationally independent subsets $\{S_i \mid i \in \omega\}$.

Take an injection $f : \omega_1 \to \mathbb{R}$. For any i put

- $m_i = 2$,
- $P(i, \alpha, y) \equiv f(\alpha) + y \in S_i$.

From CH-equivalences to CH-systems

Theorem (Erdős and Komjáth 1990)

CH holds if and only if the plane can be colored with countably many colors with no monochromatic right-angled triangle.

Take an injection $f : \omega_1 \to \mathbb{R}$. And $g : \mathbb{R} \to \omega$ a coloring which avoids right-angled triangles. For any i put

- $m_i = 2$,
- $P(i, \alpha, y) \equiv g(f(\alpha), y) = i$.

Σ^1_2-CH-systems

P is a Σ^1_2-CH-system if it is a CH-system and P is Σ^1_2.

Theorem (Addison 1959)

If $\mathbb{R} \subseteq L$ then there exists a Δ^1_2 strong well-ordering of the reals.

Theorem(Mansfield and Solovay 1970)

Let A be a $\Sigma^1_2(a)$ set. Then either $A \subseteq L[a]$, or else A contains a perfect set. In particular, if a Σ^1_2 set contains a non-constructible real then it is uncountable.

Theorem

There exists a Σ^1_2-CH-system if and only if $\mathbb{R} \subseteq L$ holds.

Summing up

What we have:

- Given a CH-system, then its Σ^1_2 version implies $\mathbb{R} \subseteq L$.
- The notion of (κ, λ)-system which is equivalent to $2^\kappa = \lambda$ for any infinite cardinals $\kappa < \lambda$.
- The notion of $\Diamond$-system which is equivalent to $\Diamond$.

What we would like to have:

- Is it consistent to have a Π^1_1-CH-system?
- Is there a $\mathsf{T} \subseteq \mathsf{ZFC}$ such that if

$$\mathsf{T} \vdash \text{ CH implies } \exists X \Phi(X),$$

then

$$\mathsf{T} \vdash \text{ CH implies } \exists X \in \Delta^1_1(\prec)\Phi(X)?$$

Summing up

What we have:

- Given a CH-system, then its Σ^1_2 version implies $\mathbb{R} \subseteq L$.
- The notion of (κ, λ)-system which is equivalent to $2^\kappa = \lambda$ for any infinite cardinals $\kappa < \lambda$.
- The notion of $\Diamond$-system which is equivalent to $\Diamond$.

What we would like to have:

- Is it consistent to have a Π^1_1-CH-system?
- Is there a $\mathsf{T} \subseteq \mathsf{ZFC}$ such that if

$$\mathsf{T} \vdash \text{CH implies } \exists X \Phi(X),$$

  then

$$\mathsf{T} \vdash \text{CH implies } \exists X \in \Delta^1_1(\prec)\Phi(X)?$$

Thank you!