Terminating via Ramsey's Theorem

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A first question

```
while ( x > 0 AND y > 0 )
      if ( y > 1 )
         ( x, y, z ) = ( z+1, y-1, z+1 )
      else
         ( x, y, z ) = ( x-1, y+1 , z+1 )
```

Does this program terminate for any x, y and z?

A first question

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while ( x > 0 AND y > 0 )
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         ( x, y, z ) = ( x-1, y+1 , z+1 )
```

Does this program terminate for any x, y and z?

NO! Indeed:

$$\langle 1,1,1\rangle \to \langle 1,2,2\rangle \to \langle 3,1,3\rangle \to \langle 2,2,4\rangle \to \langle 5,1,5\rangle \to \\ \cdots \to \langle 2n+1,1,2n+1\rangle \to \langle 2n,2,2n+2\rangle \to \dots$$

Another first question

```
while ( x > 0 AND y > 0 )
      if ( y > 1 )
         ( x, y, z ) = ( x, y-1, 2*z )
      else
         ( x, y, z ) = ( x-1, 2*z, 2*z )
```

Does this program terminate for any x, y and z?

Transition-based programs

A transition-based program $P = (S, I, R)$ consists of:

- S: a set of states,
- I: a set of initial states, such that $I \subseteq S$,
- R: a transition relation, such that $R \subseteq S \times S$.

A computation is a maximal sequence of states $s_0, \ s_2, \ \ldots$ such that

- $s_0 \in I$,
- $(s_{i+1}, s_i) \in R$ for any $i \in \mathbb{N}$.

The set Acc of accessible states is the set of all states which appear in some computation.

Termination Theorem by Podelski and Rybalchenko

- A program P is terminating if its transition relation R restricted to the accessible states is well-founded.

- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$.

- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem(Podelski and Rybalchenko 2004)

The program P is terminating if and only if there exists a disjunctively well-founded transition invariant for P.

Termination Theorem by Podelski and Rybalchenko

- A program P is terminating if its transition relation R restricted to the accessible states is well-founded.
- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$.
- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem(Podelski and Rybalchenko 2004)

R is well-founded if and only if there exist $k \in \mathbb{N}$ and k-many well-founded relations $R_0, \ldots, R_{k-1}$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$.

An answer

```
while ( x > 0 AND y > 0 )
      if ( y > 1 )
         ( x, y, z ) = ( x, y-1, 2*z )
      else
         ( x, y, z ) = ( x-1, 2*z, 2*z )
```

A transition invariant for this program is $R_1 \cup R_2$, where

$$R_1 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid y > 0 \wedge y' < y\}$$
$$R_2 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid x > 0 \wedge x' < x\}$$

Since each R_i is well-founded, then the program terminates.

A second question

```
while ( x > 0 AND y > 0 )
      if ( y > 1 )
         ( x, y, z ) = ( x, y-1, 2*z )
      else
         ( x, y, z ) = ( x-1, 2*z, 2*z )
```

How many steps before the program terminates?

Infinite Ramsey Theorem for pairs

If you have $\mathbb{N}$-many people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem(Ramsey 1930)

For any $k \in \mathbb{N}$ and for every k-coloring $c : [\mathbb{N}]^2 \to k$, there exists an infinite homogeneous set.

Complete disorder is impossible

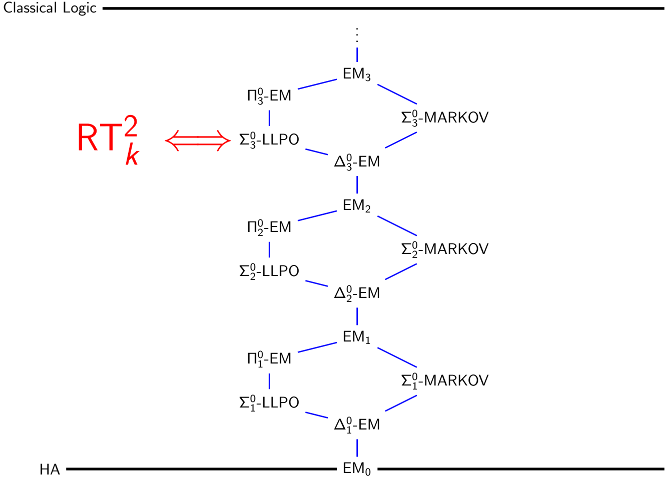
Theodore Samuel Motzkin

How many steps before the program terminates?

Hard to say, because Ramsey's Theorem is a purely classical result. Indeed,

- In 1969 Specker proved there is one recursive coloring in two colors with no recursive infinite homogeneous sets.
- In 1972 Jockusch proved that, for some recursive enumerable families of recursive colorings, it is not even possible to recursively find a color for which there is an infinite homogeneous set.

Ramsey's Theorem in the hierarchy of classical principles



H-closure Theorem

A binary relation R is H-well-founded there are no infinite decreasing transitive R-sequences.

Theorem(Berardi and S. 2014)

For any $k \in \mathbb{N}$, if $R_0, \dots, R_{k-1}$ are H-well-founded relations, then $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded.

- H-closure Theorem is classically true, because there exists a simple (i.e. within RCA_0) classical proof of the equivalence between Ramsey's Theorem and H-closure Theorem.
- By considering the inductive definition of well-foundedness, this result is intuitionistically provable.

An intuitionistic proof of the Termination Theorem

Sketch. Assume that there exists a disjunctively well-founded transition invariant, namely

$$R_0 \cup \cdots \cup R_{k-1} \supseteq R^+ \cap (\mathsf{Acc} \times \mathsf{Acc}),$$

- then R_i is H-well-founded for each $i < k$;
- hence $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded;
- therefore $R^+ \cap (\mathsf{Acc} \times \mathsf{Acc})$ is H-well-founded and transitive;
- so it is well-founded, and then also $R \cap (\mathsf{Acc} \times \mathsf{Acc})$ is.

Bounds from H-closure Theorem

A weight function for a binary relation $R \subseteq S^2$ is a function $f : S \to \mathbb{N}$ such that for any $x,\ y \in S$

$$xRy \implies f(x) < f(y).$$

$\mathcal{A}$ = the class of functions computable by a program for which there exists a disjunctively well-founded transition invariant whose relations have primitive recursive weight functions.

Proposition(Berardi, Oliva and S. 2014)

$\mathcal{A}$ =PR.

May we consider the classical definition of well-foundedness?

- Gödel's system T is simply typed λ-calculus enriched with natural numbers and primitive recursion in all finite types, together with the associated reduction rules.

- Spector's bar recursion can be intuitively explain as a recursive definition of a function through the set of the nodes of a well-founded tree.

- The Dialectica interpretation of arithmetic was extended by Spector to classical analysis in the system "T + bar recursion".

A bar recursive bound

Theorem(Berardi, Oliva and S. 2014)

Let P be a given program with transition relation R. There exists a construction Φ, definable in T+ bar recursion, such that for all $k \in \mathbb{N}$ and $\mu_0, \dots, \mu_{k-1} \colon {}^{\mathbb{N}}S \to \mathbb{N}$ and $R_0, \dots, R_{k-1}$, if

- $R^+ \subseteq R_0 \cup \dots \cup R_{k-1}$
- $\forall i < k \; \forall \sigma \; \exists j < \mu_i(\sigma) \; \neg(\sigma_{j+1} R_i \sigma_j)$

then, for all σ such that $\sigma_0 \in I$

$$\exists m < \Phi(R, \mu_0, \dots, \mu_{k-1}, R_0, \dots, R_{k-1}, \sigma) \; \neg(\sigma_{m+1} R \sigma_m).$$

Due to a result by Schwichtenberg, if $\mu_0, \dots, \mu_{k-1}$ are in system T, then also Φ is.

Might a Reverse Mathematical approach help?

- Which bounds may we get by using Reverse Mathematical tools?

- (Gasarch) Is there a natural example showing that the Termination Theorem requires the full Ramsey Theorem for pairs?

- (Gasarch) Is the Termination Theorem equivalent to Ramsey's Theorem for pairs?

Reverse Mathematics

Given a theorem of ordinary mathematics, what is the weakest subsystem of second order arithmetic in which it is provable?

- RCA_0: axioms of arithmetic, Σ^0_1-induction, Δ^0_1-comprehension.
- WKL_0: RCA_0, Σ^0_1-separation.
- ACA_0: RCA_0, arithmetical comprehension.
- ATR_0: ACA_0, Σ^1_1-separation.
- Π^1_1-CA_0: ACA_0, Π^1_1-comprehension.

Γ-induction: for any $\varphi(x)$ in Γ,

$$(\varphi(0) \wedge \forall n(\varphi(n) \implies \varphi(S(n)))) \implies \forall n \varphi(n).$$

Γ-comprehension: for any $\varphi(x)$ in Γ,

$$\exists X \forall n(n \in X \iff \varphi(n)).$$

Γ-separation: for any $\psi(x)$, $\varphi(x)$ in Γ which are exclusive,

$$\exists X \forall n(n \in X \iff \psi(n) \wedge \neg\varphi(n)).$$

Consequences of Ramsey's Theorem for pairs in two colors

- WRT^2_k. For any $c : [\mathbb{N}]^2 \to k$, there exists an infinite weakly homogeneous set; i.e. there exist $h \in k$ and $H = \{x_i : i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for any $i \in \mathbb{N}$ $c(x_i, x_{i+1}) = h$.

- CAC. Every infinite poset has an infinite chain or antichain.

- ADS. Every infinite linear ordering has an infinite ascending or descending sequence.

$$\begin{aligned} \mathsf{RCA}_0 < \mathrm{ADS} = & \mathrm{WRT}^2_2 \leq \mathrm{WRT}^2_3 \leq \dots \\ & \leq \mathrm{WRT}^2_k \leq \mathrm{CAC} < \mathrm{RT}^2_2 = \dots = \mathrm{RT}^2_k. \end{aligned}$$

The Termination Theorem in the Ramsey's zoo

- k-TT. For any relation R, if there exist $R_0, \ldots, R_{k-1}$ such that they are well-founded and $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$, then R is well-founded.

Theorem(S. and Yokoyama 2015)

- For any $k \in \mathbb{N}$, $\mathsf{RCA}_0 \vdash k\text{-}\mathrm{TT} \iff \mathrm{WRT}_k$.
- $\mathsf{RCA}_0 \vdash \forall k\ k\text{-}\mathrm{TT} \iff \forall k\ \mathrm{WRT}_k$.

Then for any $k \in \mathbb{N}$, $\mathsf{RCA}_0 \vdash \mathrm{CAC} \implies k\text{-}\mathrm{TT}$.

Answers to questions posed by Gasarch

Theorem(Hirschfeldt and Shore 2007)

CAC plus full induction does not imply RT^2_2.

Since CAC plus full induction proves $\forall k$ k-TT:

- Is there a natural example showing that the Termination Theorem requires the full Ramsey Theorem for pairs? NO!
- Is the Termination Theorem equivalent to Ramsey's Theorem for pairs? NO!

Hence, which bounds may we get by using Reverse Mathematical tools?

Weight functions and bounds

Let R be a binary relation on S.

- A weight function for R is a function $f : S \to \mathbb{N}$ such that for any $x,\ y \in S$

$$xRy \implies f(x) < f(y).$$

  We say that R has height ω if there exists a weight function for R.

  However this is not the intuitive notion of bound!

- A bound for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing sequence $\langle a_0, \ldots, a_{l-1} \rangle$, $l \leq f(a_0)$.

Weight functions vs bounds

Proposition

In RCA_0. For any relation $R \subseteq S^2$. If R has a weight function then R has a bound.

Proposition

The following are equivalent over RCA_0.

1. WKL_0.
2. For any relation $R \subseteq S^2$, R has a bound then R has a weight function.

First bounds

Theorem(Parson 1970 / Paris and Kirby 1977 / Chong, Slaman and Yang 2012)

The class of provable recursive functions of $\mathsf{WKL}_0 + \mathrm{CAC}$ is exactly the same as the class of primitive recursive functions.

Consequence

Any relation R generated by a primitive recursive transition function for which there exist k-many relations $R_0, \ldots, R_{k-1}$ with primitive recursive bounds such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ has a primitive recursive bound.

Another question

Is there a correspondence between the complexity of a primitive recursive transition bounded relation and the number of relations which compose the transition invariant?

Paris-Harrington Theorem for pairs

For given $k \in \mathbb{N}$,

- PH^{*2}_{k}: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a homogeneous set H for c such that $\min H < |H|$.

- WPH^{*2}_{k}: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a weakly homogeneous set H for c such that $\min H < |H|$.

First bounded versions of the Termination Theorem

For given $k \in \mathbb{N}$,

- k-TT^{ω}: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many relations of height ω is well-founded.

- k-TT^{b}: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many bounded relations is well-founded.

Fast Growing Hierarchy

Let F_k be the usual k-th fast growing function defined as

$$\begin{cases} F_0(x) = x + 1, \\ F_{h+1}(x) = F_h{}^{(x+1)}(x). \end{cases}$$

Let $\mathrm{Tot}(F_k)$ denote the totality of F_k:

$$\forall a \exists b (F_k(a) = b).$$

The subsystem RCA_0^*, consists of Elementary Function Arithmetic plus Δ_1^0-comprehension.

Slicing Paris-Harrington Theorem for pairs

For given $h, k \in \mathbb{N}$,

- $\mathrm{PH}_k^{h,2}$: Given $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) < F_h(f(n))$ for any $n \in \mathbb{N}$, for any coloring $c : [\mathrm{ran}(f)]^2 \to k$, there exists a homogeneous set H for c such that $\min H < |H|$.

- $\mathrm{WPH}_k^{h,2}$: Given $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) < F_h(f(n))$ for any $n \in \mathbb{N}$, for any coloring $c : [\mathrm{ran}(f)]^2 \to k$, there exists a weakly homogeneous set H for c such that $\min H < |H|$.

Hence $\mathrm{PH}^{*}{}_k^2 = \forall h\ \mathrm{PH}_k^{h,2}$ and $\mathrm{WPH}^{*}{}_k^2 = \forall h\ \mathrm{WPH}_k^{h,2}$.

H-bounds

Let R be a binary relation on S.

- A weight function for R is a function $f : S \to \mathbb{N}$ such that for any $x, \ y \in S$
$$xRy \implies f(x) < f(y).$$
We say that R has height ω if there exists a weight function for R.

- A bound for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing sequence $\langle a_0, \dots, a_{l-1} \rangle$, $l \leq f(a_0)$.

- A H-bound for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing transitive sequence $\langle a_0, \dots, a_{l-1} \rangle$, $l \leq f(a_0)$.

Bounded versions of the Termination Theorem

We say that a relation R is controlled by F_h if R is a deterministic binary relation, whose transition function $f : \mathbb{N} \to \mathbb{N}$ is such that for any $n \in \mathbb{N}$ $f(n+1) < F_h(f(n))$.

For given $k \in \mathbb{N}$,

- k-TT^h: Given R controlled by F_h, if there exists a disjunctively well-founded transition invariant for R composed of k-many relations of height ω whose weight functions f_i are such that $f_i(n) < F_h(n)$ for any n, R is well-founded.

- k-TT^h_b: Given R controlled by F_h, if there exists a disjunctively well-founded transition invariant for R composed of k-many relations with bounds f_i such that $f_i(n) < F_h(n)$ for any n, R is well-founded.

- k-TT^h_H: Given R controlled by F_h, if there exists a disjunctively well-founded transition invariant for R composed of k-many relations with H-bounds f_i such that $f_i(n) < F_h(n)$ for any n, R is well-founded.

Hence k-$\mathrm{TT}^\omega = \forall h\ k$-$\mathrm{TT}^h$ and k-$\mathrm{TT}^b = \forall h\ k$-$\mathrm{TT}^h_b$.

Bounded versions of the Termination Theorem

Theorem(S. and Yokoyama 2015)

In RCA_0^*. For any $k \in \mathbb{N}$

- $\mathrm{WPH}_k^{h,2} \Leftrightarrow k\text{-}\mathrm{TT}^h \Leftrightarrow k\text{-}\mathrm{TT}_b^h$.
- $\mathrm{PH}_k^{h,2} \Leftrightarrow k\text{-}\mathrm{TT}_H^h$.

Corollary

In RCA_0^*.

- $\forall k\ \mathrm{WPH}^{*2}_k \Leftrightarrow \forall k\ k\text{-}\mathrm{TT}^\omega \Leftrightarrow \forall k\ k\text{-}\mathrm{TT}^b$.
- $\forall k\ \mathrm{PH}^{*2}_k \Leftrightarrow \forall k\ k\text{-}\mathrm{TT}_H$.

From transition invariants to bound

Theorem(Solovay and Ketonen 1981)

In RCA_0^*. For any $k \in \mathbb{N}$, $\mathrm{Tot}(F_{k+h+5}) \implies \mathrm{PH}_k^{h,2}$.

Consequence

For any $R, R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that

- R is controlled by F_h
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$
- R_i is H-bounded by F_h for any $i \in k$

R is bounded by F_{k+h+5}.

Is it improvable?

In 2011 Figueira D., Figueira S, Schmitz and Schnoebelen observed that the Termination Theorem is a consequence of Dickson's Lemma

Theorem(Dickson 1913)

For any natural number k, every infinite sequence σ of elements in $\mathbb{N}^k$ is good; i.e. for any infinite sequence σ of elements in $\mathbb{N}^k$ there exist natural numbers $n < m$ such that $\sigma(n) \leq \sigma(m)$.

Note that given a transition-based program $P = (S, I, R)$, for which there is k-disjunctively well-founded transition invariant composed of relations of height ω we can define a map:

$$\begin{array}{rccc} \sigma : & S & \longrightarrow & \mathbb{N}^k \\ & s & \longmapsto & (f_0(s), \ldots, f_{k-1}(s)) \end{array}$$

where f_i is a weight function of R_i.

Any computation is mapped in a bad sequence!

Bounding bad sequences

Figueira D., Figueira S., Schmitz and Schnoebelen, provided a bound for the length of the bad sequences. As a corollary

Theorem (Figueira D., Figueira S., Schmitz and Schnoebelen 2011)

For any $R, R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that

- R is controlled by F_{h+1}
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$
- R_i is bounded by F_h for any $i \in k$

R is bounded by $F_{k+\max\{1,h-1\}}$.

Consequence

In RCA_0^*. $\mathrm{Tot}(F_{k+\max\{1,h-1\}}) \implies \mathrm{WPH}_k^{h+1,2}$.

And for H-bounds?

By looking for bounds via Erdős' tree.

Theorem(S. 2015)

In RCA_0^*. For any $R, R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that

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- R_i is H-bounded by F_h for any $i \in k$

R is bounded by $F_{k+\max\{1,h-1\}}$.

Consequence

In RCA_0^*. $\mathrm{Tot}(F_{k+\max\{h-1,1\}}) \implies \mathrm{PH}_k^{h+1,2}$.

Example of OPTIMAL bounds

```
while (x > 0 AND y > 0)
    if(y > 1)
                (x,y,z) = (x, y-1, 2*z)
        else    (x,y,z) = (x-1,2*z, 2*z)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$R_1 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid y > 0 \wedge y' < y\}$ Bounded by F_0

$R_2 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid x > 0 \wedge x' < x\}$ Bounded by F_0

Then R is well-founded, and R is bounded by F_{2+1}!

It is optimal since for any $x > y > 0$, the computation which starts in $(x, y, 1)$ has length greater than $F_2^x(y)$!

From bounds to transition invariants

Theorem(S. and Yokoyama 2015)

Let $k \in \mathbb{N}$. In RCA_0^* for any deterministic binary relation $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $R_0, \dots, R_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq R_0 \cup \cdots \cup R_{k+1}$ and each R_i is bounded by F_0.

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Thank you!