Reverse mathematical bounds for the Termination Theorem

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Transition-based programs

A transition-based program $P = (S, I, R)$ consists of:

- S: a set of states,
- I: a set of initial states, such that $I \subseteq S$,
- R: a transition relation, such that $R \subseteq S \times S$.

A computation is a maximal sequence of states $s_0,\ s_2,\ \ldots$ such that

- $s_0 \in I$,
- $(s_{i+1}, s_i) \in R$ for any $i \in \mathbb{N}$.

The set Acc of accessible states is the set of all states which appear in some computation.

Termination Theorem by Podelski and Rybalchenko

- A program P is terminating if its transition relation R restricted to the accessible states is well-founded.
- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^{+} \cap (\mathrm{Acc} \times \mathrm{Acc})$.
- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem(Podelski and Rybalchenko 2004)

The program P is terminating if and only if there exists a disjunctively well-founded transition invariant for P.

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Theorem(Podelski and Rybalchenko 2004)

R is well-founded if and only if there exist $k \in \mathbb{N}$ and k-many well-founded relations $R_0, \ldots, R_{k-1}$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$.

Example

```
while (x > 0 AND y > 0)
        (x,y) = (y+1, x-2)
          OR
        (x,y) = (x+2, y-2)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$$
\begin{aligned}
R_1 &:= \{(\langle x, y\rangle, \langle x', y'\rangle) \mid x + y > 0 \wedge x' + y' < x + y\} \\
R_2 &:= \{(\langle x, y\rangle, \langle x', y'\rangle) \mid y > 0 \wedge y' < y\}
\end{aligned}
.
$$

Since each R_i is well-founded, then the program terminates.

Infinite Ramsey Theorem for pairs

If you have $\mathbb{N}$-many people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem(Ramsey 1930)

For any $k \in \mathbb{N}$ and for every k-coloring $c : [\mathbb{N}]^2 \to k$, there exists an infinite homogeneous set.

Complete disorder is impossible

Theodore Samuel Motzkin

H-closure Theorem

A binary relation R is H-well-founded if any decreasing transitive R-sequence is finite.

Theorem(Berardi and S. 2014)

For any $k \in \mathbb{N}$, if $R_0, \ldots, R_{k-1}$ are H-well-founded relations, then $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded.

If we consider the inductive definition of well-foundedness, the result above is intuitionistically provable and from it we may intuitionistically prove the Termination Theorem.

Bounds from H-closure Theorem

A weight function for a binary relation $R \subseteq S^2$ is a function $f : S \to \mathbb{N}$ such that for any $x,\ y \in S$

$$xRy \implies f(x) < f(y).$$

$\mathcal{A}$ = the class of functions computable by a program for which there exists a disjunctively well-founded transition invariant whose relations have primitive recursive weight functions.

Proposition(Berardi, Oliva and S. 2014)

$\mathcal{A}$ =PR.

Reverse Mathematics

Given a theorem of ordinary mathematics, what is the weakest subsystem of second order arithmetic in which it is provable?

- RCA_0: axioms of arithmetic, Σ^0_1-induction, Δ^0_1-comprehension.
- WKL_0: RCA_0, Σ^0_1-separation.
- ACA_0: RCA_0, arithmetical comprehension.
- ATR_0: ACA_0, Σ^1_1-separation.
- Π^1_1-CA_0: ACA_0, Π^1_1-comprehension.

Γ-induction: for any $\varphi(x)$ in Γ,

$$(\varphi(0) \wedge \forall n(\varphi(n) \implies \varphi(S(n)))) \implies \forall n \varphi(n).$$

Γ-comprehension: for any $\varphi(x)$ in Γ,

$$\exists X \forall n(n \in X \iff \varphi(n)).$$

Γ-separation: for any $\psi(x)$, $\varphi(x)$ in Γ which are exclusive,

$$\exists X \forall n(n \in X \iff \psi(n) \wedge \neg\varphi(n)).$$

Which bounds may we get by using Reverse Math tools?

In 2011 Figueira D., Figueira S, Schmitz and Schnoebelen observed that the Termination Theorem is a consequence of Dickson's Lemma by the following fact:

(*) $R \subseteq \mathbb{N}^2$ is well-founded if and only if it is embedded into a well-quasi-order.

However (*) is equivalent to ACA_0 over RCA_0. Too strong for studying the strength!

Consequences of Ramsey Theorem for pairs in two colors

- WRT^2_k. For any $c : [\mathbb{N}]^2 \to k$, there exists an infinite weakly homogeneous set; i.e. there exist $h \in k$ and $H = \{x_i : i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for any $i \in \mathbb{N}$ $c(x_i, x_{i+1}) = h$.
- CAC. Every infinite poset has an infinite chain or antichain.
- ADS. Every infinite linear ordering has an infinite ascending or descending sequence.

$$\mathsf{RCA}_0 < \mathrm{ADS} = \mathrm{WRT}^2_2 \leq \mathrm{WRT}^2_3 \leq \dots$$
$$\leq \mathrm{WRT}^2_k \leq \mathrm{CAC} < \mathrm{RT}^2_2 = \dots = \mathrm{RT}^2_k.$$

The Termination Theorem in the Ramsey's zoo

- k-TT. For any relation R, if there exist $R_0, \dots, R_{k-1}$ such that they are well-founded and $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$, then R is well-founded.

Proposition

For any $k \in \mathbb{N}$:

$$\mathsf{RCA}_0 \vdash k\text{-TT} \iff \mathrm{WRT}_k.$$

Then for any $k \in \mathbb{N}$, $\mathsf{RCA}_0 \vdash \mathrm{CAC} \implies k\text{-TT}$.

Weight functions and bounds

Let R be a binary relation on S.

- A weight function for R is a function $f : S \to \mathbb{N}$ such that for any $x,\ y \in S$
$$xRy \implies f(x) < f(y).$$
We say that R has height ω if there exists a weight function for R.

However this is not the intuitive notion of bound!

- A bound for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing sequence $\langle a_0, \ldots, a_{l-1} \rangle$, $l \leq f(a_0)$.

Weight functions vs bounds

Proposition

In RCA_0. For any relation $R \subseteq S^2$. If R has a weight function then R has a bound.

Proposition

The following are equivalent over RCA_0.

1. WKL_0.
2. For any relation $R \subseteq S^2$, R has a bound then R has a weight function.

First bounds

Theorem(Parson 1970 / Paris and Kirby 1977 / Chong, Slaman and Yang 2012)

The class of provable recursive functions of $\mathsf{WKL}_0 + \mathrm{CAC}$ is exactly the same as the class of primitive recursive functions.

Consequence

Any relation R generated by a primitive recursive transition function for which there exist k-many relations $R_0, \dots, R_{k-1}$ with primitive recursive bounds such that $R_0 \cup \dots \cup R_{k-1} \supseteq R^+$ has a primitive recursive bound.

Paris-Harrington Theorem for pairs

For given $k \in \mathbb{N}$,

- PH^{*2}_k: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a homogeneous set H for c such that $\min H < |H|$.
- WPH^{*2}_k: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a weakly homogeneous set H for c such that $\min H < |H|$.

Bounded versions of the Termination Theorem

For given $k \in \mathbb{N}$,

- $k\text{-}\mathrm{TT}^{\omega}$: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many relations of height ω is well-founded.

- $k\text{-}\mathrm{TT}^{b}$: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many bounded relations is well-founded.

Proposition

In RCA_0. For any $k \in \mathbb{N}$, we have

$$\mathrm{WPH}^{*2}_{k} \Leftrightarrow k\text{-}\mathrm{TT}^{\omega} \Leftrightarrow k\text{-}\mathrm{TT}^{b}.$$

Fast growing functions

Is there a correspondence between the complexity of a primitive recursive transition bounded relation and the number of relations which compose the transition invariant?

Let F_k be the usual k-th fast growing function defined as

$$\begin{cases} F_0(x) = x + 1, \\ F_{n+1}(x) = {F_n}^{(x+1)}(x). \end{cases}$$

Sharper Bounds

Theorem(Solovay and Ketonen 1981)

In RCA_0. For any $k \in \mathbb{N}$, $\mathrm{Tot}(F_{k+4}) \implies \mathrm{PH}^{*2}_{k}$.

Consequence

For any $k, n \in \mathbb{N}$ and for any $R \subseteq \mathbb{N}^2$, R is bounded by F_{k+n+4} if there exists $R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ and each R_i is bounded by F_n.

Is it improvable?

Conjecture

For any $k, n \in \mathbb{N}$ and for any $R \subseteq \mathbb{N}^2$, R is bounded by $F_{k+\max\{n-1,1\}}$ if there exist $R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ and each R_i is bounded by F_n.

Consequence

In RCA_0. For any $k \in \mathbb{N}$, $\mathrm{Tot}(F_{k+1}) \implies \mathrm{WPH}^{*2}_k$.

Example of OPTIMAL bounds

By considering an example by Figueira D., Figueira S., Schmitz and Schnoebelen:

```
while (x > 0 AND y > 0)
    if(y > 1)
                (x,y,z) = (x, y-1, 2*z)
        else    (x,y,z) = (x-1,2*z, 2*z)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$$R_1 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid y > 0 \wedge y' < y\} \qquad \text{Bounded by } F_0$$
$$R_2 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid x > 0 \wedge x' < x\} \qquad \text{Bounded by } F_0$$

Then R is well-founded, and R is bounded by F_{2+1}!

It is optimal since for any $x > y > 0$, the computation which starts in $(x, y, 1)$ has length greater than $F_2^x(y)$!

Vice versa

Proposition

Let $k \in \mathbb{N}$. In $\mathrm{RCA}_0 + \mathrm{Tot}(F_k)$ for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $R_0, \ldots, R_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq R_0 \cup \cdots \cup R_{k+1}$ and each R_i is bounded by F_0.

Is this the minimum number of linearly bounded relations we could obtain?

Vice versa

Proposition

Let $k \in \mathbb{N}$. In $\mathrm{RCA}_0 + \mathrm{Tot}(F_k)$ for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $R_0, \ldots, R_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq R_0 \cup \cdots \cup R_{k+1}$ and each R_i is bounded by F_0.

Is this the minimum number of linearly bounded relations we could obtain?

Thank you!