Reverse mathematical bounds for the Termination Theorem

Silvia Steila
(joint work with Stefano Berardi and Keita Yokoyama)

Università degli studi di Torino

Logic Colloquium 2015
Helsinki

August 3rd - 8th, 2015

Transition-based programs

A transition-based program $P = (S, I, R)$ consists of:

- S: a set of states,
- I: a set of initial states, such that $I \subseteq S$,
- R: a transition relation, such that $R \subseteq S \times S$.

A computation is a maximal sequence of states $s_0,\ s_2,\ \ldots$ such that

- $s_0 \in I$,
- $(s_{i+1}, s_i) \in R$ for any $i \in \mathbb{N}$.

The set Acc of accessible states is the set of all states which appear in some computation.

Termination Theorem by Podelski and Rybalchenko

- A program P is terminating if its transition relation R restricted to the accessible states is well-founded.

- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$.

- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem (Podelski and Rybalchenko 2004)

The program P is terminating if and only if there exists a disjunctively well-founded transition invariant for P.

Termination Theorem by Podelski and Rybalchenko

- A program P is terminating if its transition relation R restricted to the accessible states is well-founded.

- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$.

- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem (Podelski and Rybalchenko 2004)

R is well-founded if and only if there exist $k \in \mathbb{N}$ and k-many well-founded relations $R_0, \ldots, R_{k-1}$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$.

Infinite Ramsey Theorem for pairs

If you have $\mathbb{N}$-many people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem (Ramsey 1930)

For any $k \in \mathbb{N}$ and for every k-coloring $c : [\mathbb{N}]^2 \to k$, there exists an infinite homogeneous set.

Complete disorder is impossible

Theodore Samuel Motzkin

Which bounds may we get by using Reverse Math tools?

In 2011 Figueira D., Figueira S., Schmitz and Schnoebelen observed that the Termination Theorem is a consequence of Dickson's Lemma by the following fact:

(*) $R \subseteq \mathbb{N}^2$ is well-founded if and only if it is embedded into a well-quasi-order.

However (*) is equivalent to ACA_0 over RCA_0. Too strong for studying the strength!

Consequences of Ramsey Theorem for pairs in two colors

- WRT^2_k. For any $c : [\mathbb{N}]^2 \to k$, there exists an infinite weakly homogeneous set; i.e. there exist $h \in k$ and $H = \{x_i : i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for any $i \in \mathbb{N}$ $c(x_i, x_{i+1}) = h$.

- CAC. Every infinite poset has an infinite chain or antichain.

- ADS. Every infinite linear ordering has an infinite ascending or descending sequence.

$$\begin{aligned}\mathsf{RCA}_0 < \mathrm{ADS} \leq & \mathrm{WRT}^2_2 \leq \mathrm{WRT}^2_3 \leq \dots \\ & \leq \mathrm{WRT}^2_k \leq \mathrm{CAC} < \mathrm{RT}^2_2 = \dots = \mathrm{RT}^2_k.\end{aligned}$$

The Termination Theorem in the Ramsey's zoo

- k-TT. For any relation R, if there exist $R_0, \ldots, R_{k-1}$ such that they are well-founded and $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$, then R is well-founded.

Proposition

For any $k \in \mathbb{N}$:

$$\mathsf{RCA}_0 \vdash k\text{-}\mathrm{TT} \iff \mathrm{WRT}_k.$$

Then for any $k \in \mathbb{N}$, $\mathsf{RCA}_0 \vdash \mathrm{CAC} \implies k\text{-}\mathrm{TT}$.

Which definition of bound we use?

Let R be a binary relation on S.

A weight function for R is a function $f : S \to \mathbb{N}$ such that for any $x,\ y \in S$

$$xRy \implies f(x) < f(y).$$

We say that R has height ω if there exists a weight function for R.

Definition

A bound for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing sequence $\langle a_0, \dots, a_{l-1} \rangle$, $l \leq f(a_0)$.

A H-bound for R is a function $f : S \to \mathbb{N}$ such that for any transitive R-decreasing sequence $\langle a_0, \dots, a_{l-1} \rangle$, $l \leq f(a_0)$.

First bounds

Theorem (Parson 1970 / Paris and Kirby 1977 / Chong, Slaman and Yang 2012)

The class of provable recursive functions of $\mathrm{WKL}_0 + \mathrm{CAC}$ is exactly the same as the class of primitive recursive functions.

Consequence

Any relation R generated by a primitive recursive transition function for which there exist k-many relations $R_0, \ldots, R_{k-1}$ with primitive recursive bounds such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ has a primitive recursive bound.

Paris-Harrington Theorem for pairs

For given $k \in \mathbb{N}$,

- PH^{*2}_{k}: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a homogeneous set H for c such that $\min H < |H|$.

- WPH^{*2}_{k}: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a weakly homogeneous set H for c such that $\min H < |H|$.

Bounded versions of the Termination Theorem

For given $k \in \mathbb{N}$,

$k\text{-}\mathrm{TT}^b$: any relation R for which there exists a disjunctively well-founded transition invariant composed of k-many bounded relations is well-founded.

Proposition

In RCA_0. For any $k \in \mathbb{N}$, we have

$$\mathrm{WPH}^{*2}_k \Leftrightarrow k\text{-}\mathrm{TT}^b.$$

Fast growing functions

Is there a correspondence between the complexity of a primitive recursive transition bounded relation and the number of relations which compose the transition invariant?

Let F_k be the usual k-th fast growing function defined as

$$\begin{cases} F_0(x) = x + 1, \\ F_{n+1}(x) = F_n{}^{(x+1)}(x). \end{cases}$$

Sharper Bounds

Theorem (Solovay and Ketonen 1981)

In RCA_0. For any $k \in \mathbb{N}$, $\mathrm{Tot}(F_{k+4}) \implies \mathrm{PH}^{*2}_{k}$.

Consequence

For any $k, n \in \mathbb{N}$ and for any $R \subseteq \mathbb{N}^2$, R is bounded by F_{k+n+4} if there exists $R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ and each R_i is bounded by F_n.

Is it improvable?

Conjecture

For any $k, n \in \mathbb{N}$ and for any $R \subseteq \mathbb{N}^2$, R is bounded by $F_{k+\max\{n-1,1\}}$ if there exist $R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ and each R_i is bounded by F_n.

Consequence

In RCA_0. For any $k \in \mathbb{N}$, $\mathrm{Tot}(F_{k+1}) \implies \mathrm{WPH}^{*2}_k$.

Example of OPTIMAL bounds

By considering an example by Figueira D., Figueira S., Schmitz and Schnoebelen:

```
while (x > 0 AND y > 0)
    if(y > 1)
                (x,y,z) = (x, y-1, 2*z)
        else    (x,y,z) = (x-1,2*z, 2*z)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$R_1 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid y > 0 \wedge y' < y\}$ Bounded by F_0

$R_2 := \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid x > 0 \wedge x' < x\}$ Bounded by F_0

Then R is well-founded, and R is bounded by F_{2+1}!

It is optimal since for any $x > y > 0$, the computation which starts in $(x, y, 1)$ has length greater than $F_2^x(y)$!

Vice versa

Proposition

Let $k \in \mathbb{N}$. In $\mathrm{RCA}_0 + \mathrm{Tot}(F_k)$ for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $R_0, \ldots, R_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq R_0 \cup \cdots \cup R_{k+1}$ and each R_i is bounded by F_0.

Is this the minimum number of linearly bounded relations we could obtain?

Vice versa

Proposition

Let $k \in \mathbb{N}$. In $\mathsf{RCA}_0 + \mathrm{Tot}(F_k)$ for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $R_0, \ldots, R_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq R_0 \cup \cdots \cup R_{k+1}$ and each R_i is bounded by F_0.

Is this the minimum number of linearly bounded relations we could obtain?

Thank you!