The strength of the SCT criterion

Silvia Steila

joint work with Emanuele Frittaion and Keita Yokoyama

University of Bern

JAIST
February 28th, 2017

A first informal question

Assume that a child really likes biscuits, he has z-many biscuits. Assume that his grandmother gave him x-many gold coins and y-many silver coins to buy biscuits by pursuing the following rules at each purchase:

- the child may spend one silver coin to duplicate his number of biscuits;
- the child may spend one gold coin and all his silver coins to duplicate his number of biscuits and to get one silver coin for every biscuit he has.

Does the child get infinitely many biscuits?

A first formal question

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z);
            else if (y > 1) then  c0: f( x, y-1, 2*z );
            else  c1: f( x-1, 2*z, 2*z )
```

Does this program terminate for any x, y and z?

Programs

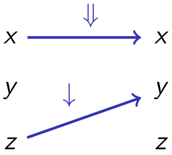
We consider a basic first order functional language:

$$
\begin{aligned}
x \in \mathrm{Par} & \quad \text{parameter identifier} \\
f \in \mathrm{Fun} & \quad \text{function identifier} \\
o \in \mathrm{Op} & \quad \text{primitive operator} \\
a \in \mathrm{AExp} & \quad \text{arithmetic expression} \\
& ::= \; x \mid x+1 \mid x-1 \mid o(a,\ldots,a) \mid f(a,\ldots,a) \\
b \in \mathrm{BExp} & \quad \text{boolean expression} \\
& ::= \; x=0 \mid x=1 \mid x<y \mid x \leq y \mid b \wedge b \mid b \vee b \mid \neg b \\
e \in \mathrm{Exp} & \quad \text{expression} \\
& ::= \; a \mid \textbf{ if } \; b \; \textbf{ then } \; e \; \textbf{ else } \; e \\
d \in \mathrm{Def} & \quad \text{function definition} \\
& ::= \; f(x_0,\ldots,x_{n-1}) = e \\
P \in \mathrm{Prog} & \quad \text{program} \\
& ::= \; d_0,\ldots,d_{m-1}
\end{aligned}
$$

Size change Graphs

Let P be a program and $f,\ g \in \mathrm{Fun}(P)$.

- A size-change graph $G : f \to g$ for P is a bipartite graph on $(\mathrm{Par}(f), \mathrm{Par}(g))$.
- The set of edges is a subset of $\mathrm{Par}(f) \times \{\downarrow, \Downarrow\} \times \mathrm{Par}(g)$ such that there is at most one edge for any $x \in \mathrm{Par}(f)$ and $y \in \mathrm{Par}(g)$.

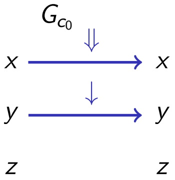


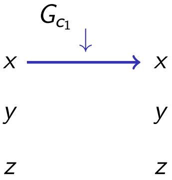
f is the source function of G and g is the target function of G.

Safe size change Graphs

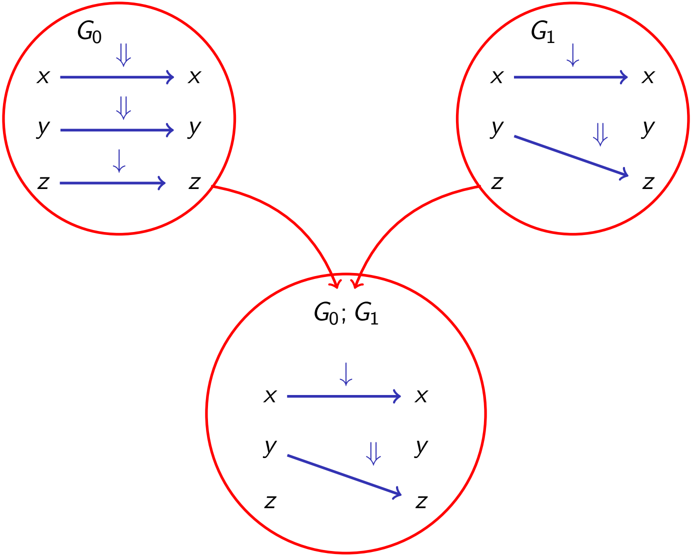
A size-change graph $G : f \to g$ for a call $\tau : f \to g$ is safe if it reflects the relationship between the parameter values in the program call.

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z);
            else if  (y > 1)  c0: f(x,y-1, 2*z)
            else   c1: f(x-1,2*z,2*z).
```





Composition of size change graphs



Multipaths and Threads

- G is idempotent if $G; G = G$.
- Given a finite set of size-change graphs $\mathcal{G}$, $\text{cl}(\mathcal{G})$ is the smallest set which contains $\mathcal{G}$ and is closed by composition.
- A multipath $\mathcal{M}$ is a sequence $G_0, \ldots, G_n, \ldots$ of graphs such that the target function of G_i is the source function of G_{i+1} for all i.
- A thread is a connected path of edges in $\mathcal{M}$ that starts at some G_t, where $t \in \mathbb{N}$.
- A multipath $\mathcal{M}$ has infinite descent if some thread in $\mathcal{M}$ contains infinitely many decreasing edges.

SCT Criterion

- A description $\mathcal{G}$ of P is a finite set of size-change graphs such that to every call $\tau : f \to g$ of P corresponds exactly one $G_\tau \in \mathcal{G}$.
- A description $\mathcal{G}$ of P is safe if each graph in $\mathcal{G}$ is safe.
- We say that a description $\mathcal{G}$ of P is size-change terminating (SCT) if every infinite multipath $\mathcal{M} = G_0, \ldots, G_n, \ldots$, where every graph $G_n \in \mathcal{G}$, has an infinite descent.

Theorem (Lee, Jones, Ben Amram 2001)

A program P is SCT iff every idempotent $G \in \mathrm{cl}(\mathcal{G}_P)$ has an arc $x \xrightarrow{\downarrow} x$.

SCT Criterion

- A description $\mathcal{G}$ of P is a finite set of size-change graphs such that to every call $\tau : f \to g$ of P corresponds exactly one $G_\tau \in \mathcal{G}$.
- A description $\mathcal{G}$ of P is safe if each graph in $\mathcal{G}$ is safe.
- We say that a description $\mathcal{G}$ of P is size-change terminating (SCT) if every infinite multipath $\mathcal{M} = G_0, \ldots, G_n, \ldots,$ where every graph $G_n \in \mathcal{G}$, has an infinite descent.

Theorem (Lee, Jones, Ben Amram 2001)

Let $\mathcal{G}$ be a description of P. Then $\mathcal{G}$ is SCT iff every idempotent $G \in \mathrm{cl}(\mathcal{G})$ has an arc $x \stackrel{\downarrow}{\to} x$.

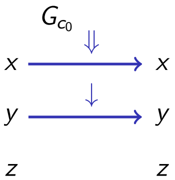
From SCT to termination

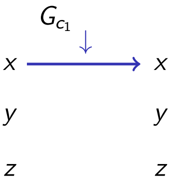
- A multipath $\mathcal{M}$ has infinite descent if some thread in $\mathcal{M}$ contains infinitely many decreasing edges.
- We say that a description $\mathcal{G}$ of P is size-change terminating (SCT) if every infinite multipath $\mathcal{M} = G_0, \ldots, G_n, \ldots,$ where every graph $G_n \in \mathcal{G}$, has an infinite descent.
- A program P with a safe SCT description does not have infinite state transition sequences.
- Thus the existence of a safe SCT description is a sufficient condition for termination.

An answer

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z);
            else if (y > 1) then  c0: f( x, y-1, 2*z );
            else  c1: f( x-1, 2*z, 2*z )
```

A safe description:





Since every idempotent $G \in \mathrm{cl}(\mathcal{G})$ has an arc $x \xrightarrow{\downarrow} x$, then the program terminates.

A second question

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z);
            else if (y > 1) then  c0: f( x, y-1, 2*z );
            else  c1: f( x-1, 2*z, 2*z )
```

Where can we prove that this program terminates?

Reverse Mathematics

Given a theorem of ordinary mathematics, what is the weakest subsystem of second order arithmetic in which it is provable?

- RCA_0: axioms of arithmetic, Σ^0_1-induction, Δ^0_1-comprehension.
- WKL_0: RCA_0, Σ^0_1-separation.
- ACA_0: RCA_0, arithmetical comprehension.
- ATR_0: ACA_0, Σ^1_1-separation.
- Π^1_1-CA_0: ACA_0, Π^1_1-comprehension.

Γ-induction: for any $\varphi(x)$ in Γ,

$$(\varphi(0) \wedge \forall n(\varphi(n) \implies \varphi(S(n)))) \implies \forall n \varphi(n).$$

Γ-comprehension: for any $\varphi(x)$ in Γ,

$$\exists X \forall n(n \in X \iff \varphi(n)).$$

Γ-separation: for any $\psi(x)$, $\varphi(x)$ in Γ which are exclusive,

$$\exists X \forall n((\psi(n) \implies n \in X) \wedge (\varphi(n) \implies n \notin X)).$$

Where can we prove that this program terminates?

- In which subsystem can we prove the SCT criterion?

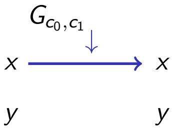
- In which subsystem can we prove that every SCT program terminates? (i.e. Soundness)

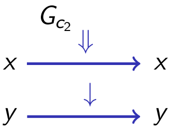
A safe description for Ackerman

```
A(x,y) := if ( x = 0) then y+1;

          else if (y = 0) then  c0: A(x-1,1);

          else  c1: A(x-1, c2: A(x,y-1))
```





Infinite Ramsey Theorem for pairs

If you have $\mathbb{N}$-many people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

(RT^1_k) For any $c\colon \mathbb{N} \to k$ there exists $i < k$ such that $c(x) = i$ for infinitely many x.

(RT^1) $\forall k \in \mathbb{N}\ \mathrm{RT}^1_k$.

(RT^2_k) For any $c : [\mathbb{N}]^2 \to k$ there exists an infinite homogeneous set $X \subseteq \mathbb{N}$, that is $c''[X]^2$ is constant.

(RT^2) $\forall k \in \mathbb{N}\ \mathrm{RT}^2_k$.

Some corollaries of Ramsey's Theorem

(Triang_k) For any coloring $c : [\mathbb{N}]^2 \to k$ there exist $i \in k$ and $t \in \mathbb{N}$ such that $c(t, m) = c(t, l) = c(m, l) = i$ for infinitely many pairs $\{m, l\}$.

(Triang) $\forall k \in \mathbb{N}\ \mathrm{Triang}_k$.

(SPP_k) For any coloring $c \colon \mathbb{N} \to k$ there exists $I \subseteq k$ such that $i \in I$ iff $i < k$ and $c(x) = i$ for infinitely many x.

(SPP) $\forall k \in \mathbb{N}\ \mathrm{SPP}_k$.

Where can we prove the SCT criterion?

Theorem (Frittaion, S., Yokoyama 2016)

Over RCA_0:

- Triang implies the SCT criterion.
- SCT criterion implies SPP.
- $\mathsf{I}\Sigma^0_2$ implies Triang.

One direction of the SCT criterion is already provable in RCA_0

Proposition (RCA_0)

Let $\mathcal{G}$ be a finite set of size-change graphs. If every multipath $M = G_0, \ldots, G_n, \ldots$ has an infinite descent, then every idempotent $G \in \mathrm{cl}(\mathcal{G})$ has an arc $x \xrightarrow{\downarrow} x$.

- Let G be idempotent.
- Then $M = G, G, \ldots, G, \ldots$ is a multipath.
- By hypothesis there exists an infinite descent.
- Define an infinite sequence $x_0, x_1, x_2, \ldots$ such that $x_i \xrightarrow{\downarrow} x_{i+1} \in G$.
- By the finite pigeonhole principle there exist $i < j$ such that $x = x_i = x_j$.
- By idempotence of G, $x \xrightarrow{\downarrow} x \in G$.

Triang implies the SCT criterion.

- Let $\mathcal{M}_\pi = G_0, \ldots, G_n, \ldots$.
- Define $c(i,j) = G_i; \ldots; G_{j-1}$.
- By $\mathrm{Triang}_{|\mathsf{cl}(\mathcal{G})|}$ to c, there exist t and $G \in \mathsf{cl}(\mathcal{G})$ such that

$$\forall n \exists m, l (n < m < l \wedge t < m \wedge c(t,m) = c(t,l) = c(m,l) = G).$$

- G is idempotent, indeed $G; G = c(t,m); c(m,l) = c(t,l) = G$.
- By hypothesis, we have that there exists $x \xrightarrow{\downarrow} x \in G$.
- We prove that there exists an infinite descent starting from x in G_t.

SCT criterion implies SPP - Case $k = 2$

- $\mathcal{G} = \{G_0, G_1, G_2\}$.
- For $i < 3$, the graph G_i has only one strict arc $z_i \xrightarrow{\downarrow} z_i$ and non-strict arcs $z_j \xrightarrow{\Downarrow} z_j$ for $j > i$.
- Define:

$$g(x) = \begin{cases} 0 & \text{if } c(x) = 0 \wedge c(x+1) = 0 \\ 1 & \text{if } c(x) = 1 \wedge c(x+1) = 1 \\ 2 & \text{otherwise} \end{cases}$$

- Consider the multipath $M = G_{g(0)}, G_{g(1)}, \ldots$ By the SCT criterion, it has an infinite descent.
- There exists a parameter z_i such that $z_i \xrightarrow{\downarrow} z_i \in G_{g(x)}$ $(g(x) = i)$ for infinitely many x.
- If $i < 2$, it means that from some point on $c(x) = i$ and so $I = \{i\}$.
- If $i = 2$ the color changes infinitely many times and so $I = \{0, 1\}$.

$\mathsf{I}\Sigma^0_2$ implies Triang

- (Slaman and Yokoyama) RT^2 is Π^1_1-conservative over $\mathsf{B}\Sigma^0_3$.
- A statement is $\widetilde{\Pi}^0_4$ if it is of the form $\forall X \varphi(X)$ and $\varphi(X) \in \Pi^0_4$.
- (e.g. Hajek and Pudlák) $\mathsf{B}\Sigma^0_3$ is $\widetilde{\Pi}^0_4$-conservative over $\mathsf{I}\Sigma^0_2$.
- Triang is $\widetilde{\Pi}^0_4$.

Where $\mathsf{B}\Sigma^0_3$ is the the bounding principle for Σ^0_3-formulas: for every σ^0_3 formula φ:

$$\forall m[(\forall i < m \exists u \varphi(i,u)) \implies \exists v(\forall i < m \exists u < v \varphi(i,u))]$$

Where can we prove soundness?

Since the SCT criterion is not provable in RCA_0:

- Let $\mathcal{G}$ a safe description of P. $\mathcal{G}$ is MSCT if it is SCT.
- Let $\mathcal{G}$ a safe description of P. $\mathcal{G}$ is ISCT if every idempotent $G \in \mathrm{cl}(\mathcal{G})$ has an arc $x \overset{\downarrow}{\to} x$.

Questions:

- If P is MSCT, then P terminates?
- If P is ISCT, then P terminates?

Work in progress

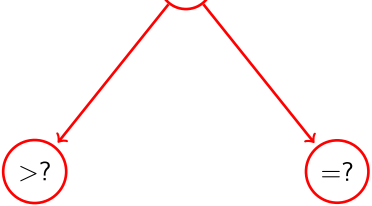
Over RCA_0:

$$\mathsf{WO}(\omega^{\omega^{\omega}}) = \text{ISCT soundness} \quad \geq \quad \text{MSCT soundness} > \mathsf{WO}(\omega^{\omega}).$$

Work in progress

Over RCA_0:

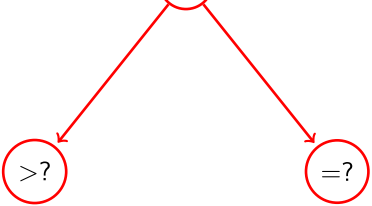
$$\mathsf{WO}(\omega^{\omega^{\omega}}) = \text{ISCT soundness} \geq \text{MSCT soundness} > \mathsf{WO}(\omega^{\omega}).$$



Work in progress

Over RCA_0:

$$\mathsf{WO}(\omega^{\omega^{\omega}}) = \text{ISCT soundness} \geq \text{MSCT soundness} > \mathsf{WO}(\omega^{\omega}).$$



Thank you!