Systems of Filters

UNIVERSITÀ DEGLI STUDI DI TORINO

Giorgio Audrito and Silvia Steila

Università degli studi di Torino

DIPARTIMENTO DI MATEMATICA GIUSEPPE PEANO UNIVERSITÀ DI TORINO

Main definitions

We introduce the notion of $\mathcal{C}$-system of filters, generalizing the standard definitions of both extenders (e.g. [2, 3]) and towers of normal ideals (e.g. [4, 5]). This provides a framework to develop the theory of extenders and towers in a more general and concise way.

Extenders

$\mathbb{E} = \left\{F_a : a \in [\lambda]^{<\omega}\right\}$ is a standard $\langle\kappa,\lambda\rangle$-extender with supports $\langle\kappa_a : a \in [\lambda]^{<\omega}\rangle$ iff the following holds.

- (Filter property) For all $a \in [\lambda]^{<\lambda}$, F_a is a κ-complete filter on $[\kappa_a]^{|a|}$ and κ_a is the least ξ such that $[\xi]^{|a|} \in F_a$.
- (Compatibility) if $a \subseteq b \in [\lambda]^{<\omega}$ then
  - $\kappa_a \leq \kappa_b$ and $\kappa_a = \kappa_b$ whenever $\max(a) = \max(b)$;
  - $A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$;
- (Uniformity) $\kappa_{\{\kappa\}} = \kappa$;
- (Normality) Assume that $a \in [\lambda]^{<\omega}$, $A \in I_a^+$, $u : A \to \kappa_a$, $i \in |a|$ are such that $u(s) \in s_i$ for all $s \in A$. Then there exist $\beta \in a_i$ and $B <_{\mathbb{E}} A$ with $B \in I_b^+$ for $b = a \cup \{\beta\}$ such that for all $s \in B$, $u(\pi_{ba}(s)) = s_j$, where $b_j = \beta$.

Towers

$\mathbb{T} = \{F_a : a \in V_\lambda\}$ is a standard tower of height λ iff the following holds.

- (Filter property) For all $a \in V_\lambda$, F_a is a non trivial filter.
- (Compatibility) For all $a \subseteq b$, $A \in F_a$ iff $\{X \subseteq b : X \cap a \in A\} \in F_b$.
- (Fineness) For all $a \in V_\lambda$ and $x \in a$ we have $\{X \subseteq a : x \in X\} \in F_a$.
- (Normality) Given $A \in I_a^+$, $u : A \to V$ such that $u(X) \in X$ for any $X \in A$, there exist $B <_{\mathbb{T}} A$, $B \in I_b^+$ and a fixed y such that $u(X \cap a) = y$ for all $X \in B$.

$$\begin{aligned} O_a &\longrightarrow \kappa_a^{|a|} \\ f &\longmapsto \operatorname{ran}(f) \end{aligned}$$

$$\mathcal{C} = [\lambda]^{<\omega}$$

$$\mathcal{C} = V_\lambda$$

$$\begin{aligned} O_a &\longrightarrow \mathcal{P}(a) \\ f &\longmapsto \operatorname{dom}(f) \end{aligned}$$

$\mathcal{C}$-Systems of Filters

We say that a set $\mathcal{C}$ is a *directed set of domains* iff $\mathcal{C}$ is closed under subsets and unions and $\bigcup\mathcal{C}$ is transitive. Given a domain $a \in V$, we define $O_a = \{\pi_M \restriction (a \cap M) : M \subseteq \operatorname{trcl}(a)\}$, where π_M is the Mostowski collapse map of M. If $a \subseteq b$, we define the *standard projection* $\pi_{ba} : O_b \to O_a$ by $\pi_{ba}(f) = f \restriction a$.
Define $x \trianglelefteq y$ as $x \in y \vee x = y$. We say that $u : O_a \to V$ is *regressive* on $A \subseteq O_a$ iff for all $f \in A$, $u(f) \trianglelefteq f(x_f)$ for some $x_f \in \operatorname{dom}(f)$. We say that u is *guessed* on $B \subseteq O_b$, $b \supseteq a$ iff there is a fixed $y \in b$ such that for all $f \in B$, $u(\pi_{ba}(f)) = f(y)$.
Let $V \subseteq W$ be transitive models of ZFC and $\mathcal{C} \in V$ be a directed set of domains.
We say that $\mathbb{S} = \{F_a : a \in \mathcal{C}\}$ is a $\mathcal{C}$-system of filters with respect to V iff the following holds:

- *(Filter property)* for all $a \in \mathcal{C}$, $F_a \subseteq \mathcal{P}^V(O_a)$ is a non-trivial filter;
- *(Fineness)* for all $a \in \mathcal{C}$ and $x \in a$, $\{f \in O_a : x \in \operatorname{dom}(f)\} \in F_a$;
- *(Compatibility)* for all $a \subseteq b$ in $\mathcal{C}$ and $A \subseteq O_a$, $A \in F_a \iff \pi_{ba}^{-1}[A] \in F_b$;
- *(Normality)* every function that is regressive on a set $A \in I_a^+$ for some $a \in \mathcal{C}$ is guessed on a set $B \in I_b^+$ for some $b \in \mathcal{C}$ such that $B \subseteq \pi_b^{-1}[A]$;

We say that $\mathcal{S}$ is a $\mathcal{C}$-system of ultrafilters if in addition:

- *(Ultrafilter)* for all $a \in \mathcal{C}$, F_a is an ultrafilter.

Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters, a be in $\mathcal{C}$. We say that κ_a is the *support* of a iff it is the minimum α such that $O_a \cap {}^aV_\alpha \in F_a$. We say that $\mathbb{S}$ is a $\langle\kappa,\lambda\rangle$-system of filters if and only if:

- it has length λ and $\kappa \subseteq \bigcup\mathcal{C}$,
- $F_{\{\gamma\}}$ is principal generated by $\operatorname{id} \restriction \{\gamma\}$ whenever $\gamma < \kappa$,
- $\kappa_a \leq \kappa$ whenever $a \in V_{\kappa+1}$.

Elementary embedding derived from a system of ultrafilters

Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters, and define

$$U_{\mathcal{S}} = \{u : O_a \to V \ : \ a \in \mathcal{C}\}$$

and the relations

$$u =_{\mathcal{S}} v \Leftrightarrow \{f \in O_c : \ u(\pi_{ca}(f)) = v(\pi_{cb}(f))\} \in F_c$$
$$u \in_{\mathcal{S}} v \Leftrightarrow \{f \in O_c : \ u(\pi_{ca}(f)) \in v(\pi_{cb}(f))\} \in F_c$$

where $O_a = \operatorname{dom}(u)$, $O_b = \operatorname{dom}(v)$, $c = a \cup b$. The ultrapower of V by $\mathcal{S}$ is $\operatorname{Ult}(V,\mathcal{S}) = \langle U_{\mathcal{S}}/=_{\mathcal{S}}, \ \in_{\mathcal{S}}\rangle$.
Define $j_{\mathcal{S}} : V \to \operatorname{Ult}(V,\mathcal{S})$ by $j_{\mathcal{S}}(x) = [c_x]_{\mathcal{S}}$, $c_x : O_\emptyset \to \{x\}$. We can prove Łoś Theorem for $\operatorname{Ult}(V,\mathcal{S})$ hence that the map $j_{\mathcal{S}}$ is elementary. Furthermore, we can study the structure of the ultrapower via the following canonical representatives:

- $[c_x]_{\mathcal{S}} = j_{\mathcal{S}}(x)$ for any $x \in V$;
- $[\operatorname{proj}_x]_{\mathcal{S}} = x$ for any $x \in \bigcup\mathcal{C}$, where

$$\begin{aligned} \operatorname{proj}_x : O_{\{x\}} &\longrightarrow V \\ f &\longmapsto f(x) \end{aligned}$$

- $[\operatorname{ran}_a]_{\mathcal{S}} = a$ for any $a \in \mathcal{C}$;
- $[\operatorname{dom}_a]_{\mathcal{S}} = j_{\mathcal{S}}[a]$ for any $a \in \mathcal{C}$;
- $[\operatorname{id}_a]_{\mathcal{S}} = (j_{\mathcal{S}} \restriction a)^{-1}$ for any $a \in \mathcal{C}$.

System of ultrafilters derived from an elementary embedding

Let $j : V \to M \subseteq V[G]$ be a generic elementary embedding, $\mathcal{C} \in V$ be a directed set of domains such that $(j \restriction a)^{-1} \in M$ for all $a \in \mathcal{C}$. The $\mathcal{C}$-*system of ultrafilters derived from* j is $\mathcal{S} = \langle F_a : a \in \mathcal{C}\rangle$ such that:

$$F_a = \left\{A \subseteq O_a : (j \restriction a)^{-1} \in j(A)\right\}.$$

Given a $\mathcal{C}$-system of ultrafilters $\mathcal{S}$, the $\mathcal{C}$-system of ultrafilters derived from $j_{\mathcal{S}}$ is $\mathcal{S}$ itself. We can also define the *factor map associated to* $\mathcal{S}$ as:

$$\begin{aligned} k : \quad \operatorname{Ult}(V,\mathcal{S}) &\longrightarrow M \\ [u : O_a \to V]_{\mathcal{S}} &\longmapsto j(u)((j \restriction a)^{-1}) \end{aligned}$$

that is elementary, has $\operatorname{crit}(k) \geq \lambda$, and commutes with $j_{\mathcal{S}}$, j.

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Systems of filters in V and generic systems of ultrafilters

Let $\dot F$ be a $\mathbb{B}$-name for an ultrafilter on $\mathcal{P}^V(X)$. Define

$$\mathbf{I}(\dot F) = \left\{Y \subset X : \ \left[\!\left[\check Y \in \dot F\right]\!\right] = \mathbf{0}_{\mathbb{B}}\right\}$$

Let I be an ideal in V on $\mathcal{P}(X)$ and consider the poset $\mathbb{B} = \mathcal{P}(X)/I$. Let $\dot{\mathbf{F}}(I)$ be the $\mathbb{B}$-name defined by

$$\dot{\mathbf{F}}(I) = \left\{\langle\check Y, [Y]_I\rangle : \ Y \subseteq X\right\}$$

Notice that $\mathbf{I}(\dot{\mathbf{F}}(I)) = I$, while there might not be any relation between $\dot{\mathbf{F}}(\mathbf{I}(\dot F))$ and $\dot F$. Let $\dot F$ be a $\mathbb{B}$-name for an ultrafilter on $\mathcal{P}^V(X)$ and set $\mathbb{C} = \mathcal{P}(X)/\mathbf{I}(\dot F)$. Then the *immersion* map $i_{\dot F}$ defined as follows is a (incomplete) morphism of cBas:

$$\begin{aligned} i_{\dot F} : \quad \mathbb{C} &\longrightarrow \mathbb{B} \\ [A] &\longmapsto \left[\!\left[\check A \in \dot F\right]\!\right]_{\mathbb{B}} \end{aligned}$$

Let $\dot{\mathcal{S}} = \langle\dot F_a : \ a \in \mathcal{C}\rangle$ be a $\mathbb{B}$-name for a $\mathcal{C}$-system of ultrafilters. Then $\mathbf{I}(\dot{\mathcal{S}}) = \langle I_a = \mathbf{I}(\dot F_a) : \ a \in \mathcal{C}\rangle$ is the corresponding system of filters in V. Conversely, let $\mathbb{S} = \langle I_a : \ a \in \mathcal{C}\rangle$ be a $\mathcal{C}$-system of filters in V. Then $\dot{\mathbf{F}}(\mathbb{S}) = \langle\dot F_a = \dot{\mathbf{F}}(I_a) : \ a \in \mathcal{C}\rangle$ is the corresponding name for a system of ultrafilters.

Systems of ultrafilters and Lévy collapse

Let $\mathbb{S}$ be a $\langle\kappa,\lambda\rangle$-system of filters, $\mathbb{C}$ be a κ-cc cBa. Define $\mathbb{S}^{\mathbb{C}} = \left\{F_s^{\mathbb{C}} : \ s \in \mathcal{C}\right\}$ where

$$F_s^{\mathbb{C}} = \left\{A \subseteq (O_a)^{V^{\mathbb{C}}} : \ \exists B \in \check F_s \ A \supseteq B\right\}$$

Then $\mathbb{S}^{\mathbb{C}}$ is a $\mathcal{C}$-system of filters, $\mathbb{C} * \mathbb{S}^{\mathbb{C}}$ is isomorphic to $\mathbb{S} * j(\mathbb{C})$ and the following diagram commutes (where $j^{\mathbb{C}}$ is the embedding induced by $\mathbb{S}^{\mathbb{C}}$ and the vertical inclusions are forcing extensions):

$$\begin{array}{ccccccc} V & \xrightarrow{j_{\dot{\mathbf{F}}(\mathbb{S})}} & M & \subseteq & V^{\mathbb{S}} & & \\ \cap\!\!\!\!| & & \cap\!\!\!\!| & & \cap\!\!\!\!| & & \\ V^{\mathbb{C}} & \xrightarrow[j_{\dot{\mathbf{F}}(\mathbb{S}^{\mathbb{C}})}]{} & M^{j(\mathbb{C})} & \subseteq & V^{\mathbb{S}*j(\mathbb{C})} & = & V^{\mathbb{C}*\mathbb{S}^{\mathbb{C}}} \end{array}$$

This fact can be used to collapse a generic large cardinal while maintaining its large cardinal properties.

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