# Ramsey Theorem for pairs as a classical principle in Intuitionistic Arithmetic.

Silvia Steila,
joint work with Stefano Berardi

Università degli studi di Torino

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## $\Sigma^0_n$-LLPO

*Lesser Limited Principle of Omniscience. For any parameter a*

$$\forall x, x' \, (P(x,a) \vee Q(x',a)) \implies \forall x P(x,a) \vee \forall x Q(x,a). \, (P,\ Q \in \Sigma^0_{n-1})$$

## Pigeonhole Principle for $\Pi^0_n$

*For any parameter a*

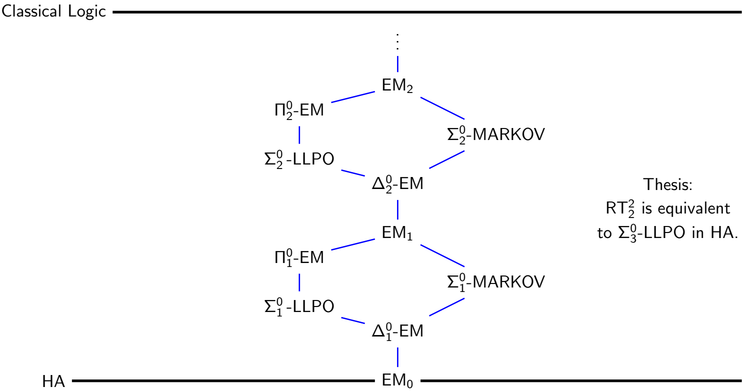
$$\forall x \, \exists z \, [z \geq x \ \wedge \ (P(z,a) \vee Q(z,a))] \implies$$

$$\forall x \, \exists z \, [z \geq x \ \wedge \ P(z,a)] \ \vee \ \forall x \, \exists z \, [z \geq x \ \wedge \ Q(z,a)]. \, (P, Q \in \Pi^0_n)$$

## $EM_n$

*Excluded Middle for $\Sigma^0_n$ formulas. For any parameter a*

$$\exists x \ P(x,a) \ \vee \ \neg\exists x \ P(x,a). \, (P \in \Pi^0_{n-1})$$




Thesis:
$RT^2_2$ is equivalent
to $\Sigma^0_3$-LLPO in HA.

The purpose of this work is to study, from the viewpoint of first order arithmetic (no set variables, the only sets are the arithmetical sets), Ramsey Theorem for pairs for recursive assignement of two colors in order to find some principle of classical logic equivalent to it in HA.

If $X$ is a set,

$$[X]^2 = \{Y \subseteq X \mid |Y| = 2\}.$$

We can think of $[X]^2$ as the complete graph on $X$. We only consider arithmetically definable sets.

**$\mathbf{RT^2_2(\Sigma^0_n)}$.** Ramsey Theorem for graphs and $\Sigma^0_n$ 2-colorings

*For any coloring $c_a : [\omega]^2 \to 2$ with a parameter $a$, there exists an infinite subset $X$ of $\omega$ homogeneous for the given coloring, i.e. $[X]^2$ is painted with only one color. ($c_a \in \Sigma^0_n$).*

In this work we formalize Ramsey Theorem for two colors, for pairs and for recursive colorings by the following schema:

$$\{\forall a(B(., c_a) \text{ infinite homogeneous black } \vee W(., c_a) \text{ infinite homogeneous white }) \mid \text{ for some } B, W \text{ arithmetical predicates }\}.$$

Here $c = \{c_a \mid a \in \omega\}$ denotes any recursive family of recursive assignment of two colors, black and white. We call this a disjunctive schema and we prove it if we prove some instance of it.

# First part: $\mathrm{RT}^2_2$ implies $\Sigma^0_3$-LLPO

We will prove that in the first order intuitionistic arithmetic, $\mathrm{RT}^2_2(\Sigma^0_0)$ is equivalent to the classical principle $\Sigma^0_3$-LLPO. By definition of disjunctive schema, Ramsey Theorem for graphs and a recursive 2-coloring implies $\Sigma^0_3$-LLPO if the following holds: for each $P$ in $\Sigma^0_3$-LLPO, there exist a finite number of recursive family of recursive colorings $c_{a,0}, \ldots, c_{a,j-1}$ such that, fixed any $W_i(., c_{a,i})$ and $B_i(., c_{a,i})$, if we assume

$$\{\forall a(W_i(., c_{a,i}) \text{ is inf. and hom. } \vee B_i(., c_{a,i}) \text{ is inf. and hom. }) \mid i \in j\}$$

then we deduce $P$.

# Proof sketch of $\mathrm{RT}^2_2$ implies $\Sigma^0_3$-LLPO

**Lemma**

1. $\mathrm{RT}^2_2(\Sigma^0_0)$ *implies* $\mathrm{EM}_1$*;*
2. $\mathrm{EM}_1$ *implies that, for any family* $F = \{s(n,\cdot) \mid n \in \omega\}$ *of recursive monotone and bounded above sequences enumerated by a binary primitive recursive function* $s : \omega \times \omega \to \omega$*, each sequence in* $F$ *is stationary;*
3. $\mathrm{EM}_1$ *implies that, for any family* $G = \{t(n,\cdot) \mid n \in \omega\}$ *of recursive sequences enumerated by a binary primitive recursive function* $t : \omega \times \omega \to \omega$ *for which there are at most* $k$ *values of* $x$ *such that* $t(n,x) \neq t(n,x+1)$*, each sequence in* $G$ *is stationary.*

# Proof sketch of $RT^2_2$ implies $\Sigma^0_3$-LLPO

Let $a$ be a parameter. We assume the hypothesis of $\Sigma^0_3$-LLPO:

$$\forall x,\ x' \ (H_0(x, a) \vee H_1(x', a)),$$

where

$$H_0(x, a) := \exists y \ \forall z \ P_0(x, y, z, a)$$
$$H_1(x, a) := \exists y \ \forall z \ P_1(x, y, z, a)$$

for some $P_0,\ P_1$ primitive recursive predicates.

# Proof sketch of $RT^2_2$ implies $\Sigma^0_3$-LLPO

Our thesis is

$$\forall x\ H_0(x,a) \vee \forall x\ H_1(x,a)$$

we define a recursive 2-coloring such that:

- if there are infinitely many white edges from $x$, then for all $y \leq x$ $H_0(y,a)$ holds;
- if there are infinitely many black edges from $x$, then for all $y \leq x$ $H_1(y,a)$ holds.

Applying $RT^2_2(\Sigma^0_0)$, there exists an infinite homogeneous set $X$. We prove that if there is some infinite set $X$ homogeneous in color $c$, then $H_c(x,a)$ holds for infinitely many $x$. We obtain

$$\forall x\ H_c(x,a).$$

# Second part: $\Sigma^0_3$-LLPO implies $\mathrm{RT}^2_2$

Now we modify Jockusch proof of Ramsey Theorem in order to obtain a proof in HA of $\Sigma^0_3$-LLPO $\implies$ $\mathrm{RT}^2_2$.

Given a coloring $c : [\omega]^2 \to 2$ we say that $X \subseteq \omega$ defines a 1-coloring if for all $x \in X$, any two edges from $x$ to some $y, z \in X$ have the same color.

If $X$ is infinite and defines a 1-coloring, thanks to the Pigeonhole Principle we may define an infinite arithmetical subset $Y$ of $X$ whose points all have the same color. $Y$ is homogeneous for $c$.

So we need to find an infinite set that defines a 1-coloring.

# Proof Sketch of $\Sigma^0_3$-LLPO implies $RT^2_2$

A tree $T$ included in a graph $\omega$ defines a 1-coloring w.r.t. $T$ if for all $x \in T$ for any two proper descendants $y,\ z$ of $x$ in $T$, the edges from $x$ to $y,\ z$ have the same color.

Assume there exists some infinite binary tree $T$ defining a 1-coloring w.r.t. $T$. Then $T$ has an infinite branch $B$ by König's Lemma. $B$ defines an infinite 1-coloring and so proves $RT^2_2$.

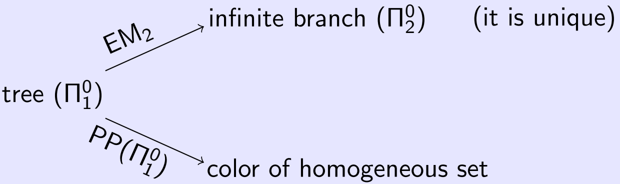
Therefore a sufficient condition for $RT^2_2$ is the existence of an infinite binary tree defining a 1-coloring.

# Proof Sketch of $\Sigma^0_3$-LLPO implies $RT^2_2$

Jockusch proof:

$$\text{tree} \xrightarrow{\text{König}} \text{infinite branch} \xrightarrow{PP(\Pi^0_2)} \text{homogeneous set}$$

Our work ($\Sigma^0_3$-LLPO):



# Proof Sketch of $\Sigma^0_3$-LLPO implies $RT^2_2$

Given any set $T$ we may equip it by the following ancestor/descendant relation

- $0 \prec_T 1$,
- $x \prec_T y$ iff $x \in T$ , $y \in \omega$, $x < y$ and $\forall z(z \prec_T x(c(\{z, x\} = c\{z, y\})))$.

**Definition (Inductive definition of the set $T$ in HA)**

Define $T_n$ by induction on $n$.

- If $n = 0$ then $T_0 = x_0 := 0$.
- For $n + 1$, if Chosen$(x_{n+1}, T_n)$, then $T_{n+1} = T_n \cup \{x_{n+1}\}$.

$$T = \bigcup_{n \in \omega} T_n.$$

Chosen is some suitable arithmetical predicate.

# Proof Sketch of $\Sigma^0_3$-LLPO implies $\mathrm{RT}^2_2$

We proved (using a part of $\Sigma^0_3$-LLPO) that

- $T$ is a $\Pi^0_1$ binary tree,
- $T$ has a unique infinite branch $r$ such that if $T$ has infinitely many edges with color $c$, then $r$ has infinitely many edges with color $c$.

Moreover using

$$\Sigma^0_3\text{-LLPO} \implies \text{Pigeonhole Principle for } \Pi^0_1 \text{ predicates },$$

we obtain that $T$ has infinitely many edges of color $c$, so $r$ has infinite many edges of color $c$; their smaller nodes define a monochromatic set for the original graph.

Our proof recursively defines two monochromatic sets $\Delta^0_3$, one of each color, that can not be both finite, even if we can not decide which of these is the infinite one.

In Jockusch proof he shows that one of the homogeneous sets is $\Pi^0_2$, while the second one in $\Delta^0_3$. In our proof we can see that both the homogeneous sets are $\Delta^0_3$, since our construction is symmetric with respect to the two colors.

# Conclusions

Apparently, $\Sigma^0_3$-LLPO is a principle of uncommon use, so we want to remark the equivalence between $\Sigma^0_3$-LLPO and two more common principles: $EM_2$ and DeMorgan($\Sigma^0_3$).

$$\text{DeMorgan}(\Sigma^0_{\text{n}}) := \neg(P \wedge Q) \implies \neg P \vee \neg Q. \ (P, Q \in \Sigma^0_n)$$

**Theorem (in intuitionistic arithmetic HA)**

$\Sigma^0_n$-LLPO $\iff$ DeMorgan$(\Sigma^0_{\text{n}}) + EM_{n-1}$.

We can see that the most of the proof uses only $EM_2$ and that DeMorgan($\Sigma^0_3$) (and so $\Sigma^0_3$-LLPO) is used only in the last part.

# Questions and future developments

- Is $\Sigma^0_3$-LLPO equivalent to Ramsey Theorem for graphs and recursive $n$-colorings for any $n \geq 2$?
- May we generalize our results to $\Sigma^0_n$-colorings and the classical principle $\Sigma^0_{n+3}$-LLPO, for any $n \geq 0$?
- We hope to apply, in future works, the method called interactive realizability to understand and explain the computational content of the modified Jockusch proof, and prove (constructively) some consequences of Ramsey Theory. The interactive realizability is a realizability interpretation for first order classical arithmetic introduced in 2008 by Stefano Berardi and Ugo de' Liguoro.

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