

Reverse mathematical bounds for the Termination Theorem

Silvia Steila

(joint work with Stefano Berardi and Keita Yokoyama)

Università degli studi di Torino

Logic and Information ST 2015:
Münchenwiler Meeting

March 25th - 26th, 2015

Transition-based programs

A **transition-based program** $P = (S, I, R)$ consists of:

- ▶ S : a set of **states**,
- ▶ I : a set of **initial states**, such that $I \subseteq S$,
- ▶ R : a **transition relation**, such that $R \subseteq S \times S$.

A **computation** is a maximal sequence of states s_0, s_2, \dots such that

- ▶ $s_0 \in I$,
- ▶ $(s_{i+1}, s_i) \in R$ for any $i \in \mathbb{N}$.

The set Acc of **accessible states** is the set of all states which appear in some computation.

Termination Theorem by Podelski and Rybalchenko

- ▶ A program P is **terminating** if its transition relation R restricted to the accessible states is well-founded.
- ▶ A **transition invariant** of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\text{Acc} \times \text{Acc})$.
- ▶ A relation is **disjunctively well-founded** if it is a finite union of well-founded relations.

Theorem(Podelski and Rybalchenko 2004)

The program P is terminating if and only if there exists a disjunctively well-founded transition invariant for P .

Termination Theorem by Podelski and Rybalchenko

- ▶ A program P is **terminating** if its transition relation R restricted to the accessible states is well-founded.
- ▶ A **transition invariant** of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\text{Acc} \times \text{Acc})$.
- ▶ A relation is **disjunctively well-founded** if it is a finite union of well-founded relations.

Theorem(Podelski and Rybalchenko 2004)

R is well-founded if and only if there exist $k \in \mathbb{N}$ and k -many well-founded relations R_0, \dots, R_{k-1} such that $R_0 \cup \dots \cup R_{k-1} \supseteq R^+$.

Example

```
while (x > 0 AND y > 0)
  (x,y) = (y+1, x-2)
  OR
  (x,y) = (x+2, y-2)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$$R_1 := \{(\langle x, y \rangle, \langle x', y' \rangle) \mid x + y > 0 \wedge x' + y' < x + y\}$$

$$R_2 := \{(\langle x, y \rangle, \langle x', y' \rangle) \mid y > 0 \wedge y' < y\}$$

Since each R_i is well-founded, then the program **terminates**.

Infinite Ramsey Theorem for pairs

If you have \mathbb{N} -many people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem(Ramsey 1930)

For any $k \in \mathbb{N}$ and for every k -coloring $c : [\mathbb{N}]^2 \rightarrow k$, there exists an infinite **homogeneous** set.

Complete disorder is impossible

Theodore Samuel Motzkin

H-closure Theorem

A binary relation R is **H-well-founded** if any decreasing transitive R -chain is finite.

Theorem(Berardi and S. 2014)

For any $k \in \mathbb{N}$, if R_0, \dots, R_{k-1} are H -well-founded relations, then $R_0 \cup \dots \cup R_{k-1}$ is H -well-founded.

We studied it since it is **intuitionistically provable** and from it we may **intuitionistically** prove the Termination Theorem.

Bounds from H-closure Theorem

A **weight function** for a binary relation $R \subseteq S^2$ is a function $f : S \rightarrow \mathbb{N}$ such that for any $x, y \in S$

$$xRy \implies f(x) < f(y).$$

\mathcal{A} = the class of functions computable by a program for which there exists a disjunctively well-founded transition invariant whose relations have **primitive recursive weight functions**.

Proposition(Berardi, Oliva and S. 2014)

$\mathcal{A} = \text{PR}$.

Which bounds may we get by using Reverse Math tools?

In 2011 Figueira D., Figueira S, Schmitz and Schnoebelen observed that the Termination Theorem is a consequence of **Dickson's Lemma** by the following fact:

(*) $R \subseteq \mathbb{N}^2$ is well-founded if and only if it is embedded into a well-quasi-order.

However (*) is equivalent to ACA_0 over RCA_0 . Too **strong** for studying the strength!

Consequences of Ramsey Theorem for pairs in two colors

- ▶ WRT_k^2 . For any $c : [\mathbb{N}]^2 \rightarrow k$, there exists an infinite **weakly homogeneous** set; i.e. there exist $h \in k$ and $H = \{x_i : i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for any $i \in \mathbb{N}$ $c(x_i, x_{i+1}) = h$.
- ▶ **CAC**. Every infinite poset has an infinite chain or antichain.
- ▶ **ADS**. Every infinite linear ordering has an infinite ascending or descending sequence.

$$\begin{aligned} \text{RCA}_0 &< \text{ADS} \leq \text{WRT}_2^2 \leq \text{WRT}_3^2 \leq \dots \\ &\leq \text{WRT}_k^2 \leq \text{CAC} < \text{RT}_2^2 = \dots = \text{RT}_k^2. \end{aligned}$$

The Termination Theorem in the Ramsey's zoo

- ▶ k -TT. For any relation R , if there exist R_0, \dots, R_{k-1} such that they are well-founded and $R_0 \cup \dots \cup R_{k-1} \supseteq R^+$, then R is well-founded.

Proposition

For any $k \in \mathbb{N}$:

$$\text{RCA}_0 \vdash k\text{-TT} \iff \text{WRT}_k.$$

Then for any $k \in \mathbb{N}$, $\text{RCA}_0 \vdash \text{CAC} \implies k\text{-TT}$.

Weight functions and bounds

Let R be a binary relation on S .

- ▶ A **weight function** for R is a function $f : S \rightarrow \mathbb{N}$ such that for any $x, y \in S$

$$xRy \implies f(x) < f(y).$$

We say that R has **height** ω if there exists a weight function for R .

However this is not the **intuitive** notion of bound!

- ▶ A **bound** for R is a function $f : S \rightarrow \mathbb{N}$ such that for any R -decreasing sequence $\langle a_0, \dots, a_{l-1} \rangle$, $l \leq f(a_0)$.

Weight functions vs bounds

Proposition

In RCA_0 . For any relation $R \subseteq S^2$. If R has a weight function then R has a bound.

Proposition

The following are equivalent over RCA_0 .

1. WKL_0 .
2. For any relation $R \subseteq S^2$, R has a bound then R has a weight function.

First bounds

Theorem(Parson 1970 / Paris and Kirby 1977 / Chong, Slaman and Yang 2012)

The class of provable recursive functions of $WKL_0 + CAC$ is exactly the same as the class of primitive recursive functions.

Consequence

Any relation R generated by a primitive recursive transition function for which there exist k -many relations R_0, \dots, R_{k-1} with primitive recursive bounds such that $R_0 \cup \dots \cup R_{k-1} \supseteq R^+$ has a primitive recursive bound.

Paris-Harrington Theorem for pairs

For given $k \in \mathbb{N}$,

- ▶ PH_k^{*2} : for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \rightarrow k$, there exists a **homogeneous** set H for c such that $\min H < |H|$.
- ▶ WPH_k^{*2} : for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \rightarrow k$, there exists a **weakly homogeneous** set H for c such that $\min H < |H|$.

Bounded versions of the Termination Theorem

For given $k \in \mathbb{N}$,

- ▶ $k\text{-TT}^\omega$: any relation R for which there exists a disjointively well-founded transition invariant composed of k -many relations of height ω is well-founded.

- ▶ $k\text{-TT}^b$: any relation R for which there exists a disjointively well-founded transition invariant composed of k -many bounded relations is well-founded.

Proposition

In RCA_0 . For any $k \in \mathbb{N}$, we have

$$\text{WPH}_k^{*2} \Leftrightarrow k\text{-TT}^\omega \Leftrightarrow k\text{-TT}^b.$$

Fast growing functions

Is there a correspondence between the **complexity** of a primitive recursive transition bounded relation and the **number of relations** which compose the transition invariant?

Let F_k be the usual k -th **fast growing function** defined as

$$\begin{cases} F_0(x) = x + 1, \\ F_{n+1}(x) = F_n^{(x+1)}(x). \end{cases}$$

Sharper Bounds

Theorem(Solovay and Ketonen 1981)

In RCA_0 . For any $k \in \mathbb{N}$, $\text{Tot}(F_{k+4}) \implies \text{PH}_k^{*2}$.

Consequence

For any $k, n \in \mathbb{N}$ and for any $R \subseteq \mathbb{N}^2$, R is bounded by F_{k+n+4} if there exists $R_0, \dots, R_{k-1} \subseteq \mathbb{N}^2$ such that $R_0 \cup \dots \cup R_{k-1} \supseteq R^+$ and each R_i is bounded by F_n .

Is it improvable?

Conjecture

For any $k, n \in \mathbb{N}$ and for any $R \subseteq \mathbb{N}^2$, R is bounded by $F_{k+\max\{n,2\}}$ if there exist $R_0, \dots, R_{k-1} \subseteq \mathbb{N}^2$ such that $R_0 \cup \dots \cup R_{k-1} \supseteq R^+$ and each R_i is bounded by F_n .

Conjecture

In RCA_0 . For any $k \in \mathbb{N}$, $\text{Tot}(F_{k+2}) \implies \text{WPH}_k^{*2}$.

Example of HUGE bounds

```
while (x > 0 AND y > 0)
  if(x > y)
    (x,y) = (y, x)
  else
    (x,y) = (x, y-1)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$R_1 := \{(\langle x, y \rangle, \langle x', y' \rangle) \mid x > 0 \wedge x' < x\}$ Bounded by F_0

$R_2 := \{(\langle x, y \rangle, \langle x', y' \rangle) \mid y > 0 \wedge y' < y\}$ Bounded by F_0

Then R is well-founded, it is bounded by F_6 ... or hopefully by F_4 .

Vice versa

Proposition

Let $k \in \mathbb{N}$. In $\text{RCA}_0 + \text{Tot}(F_k)$ for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $R_0, \dots, R_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq R_0 \cup \dots \cup R_{k+1}$ and each R_i is bounded by F_0 .

Is this the minimum number of linearly bounded relations we could obtain?

Vice versa

Proposition

Let $k \in \mathbb{N}$. In $\text{RCA}_0 + \text{Tot}(F_k)$ for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $R_0, \dots, R_{k+1} \subseteq \mathbb{N}^2$ such that $R^+ \subseteq R_0 \cup \dots \cup R_{k+1}$ and each R_i is bounded by F_0 .

Is this the minimum number of linearly bounded relations we could obtain?

Thank you!