

From equivalent forms of CH to CH-systems

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Törnquist and Weiss idea

In 2012 Törnquist and Weiss studied many Σ_2^1 definable version of some statements equivalent to CH ($2^{\aleph_0} = \aleph_1$).

CH \iff there exist some **objects** such that **something happens**.

They proved that these Σ_2^1 counterparts become equivalent to the statement “all reals are constructible”.

$\mathbb{R} \subseteq L$ \iff there exist some Σ_2^1 **objects** such that **something happens**.

From CH-equivalences to $(\mathbb{R} \subseteq L)$ -equivalences

Theorem (Sierpinski 1965)

CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there are Σ_2^1 sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

From CH-equivalences to $(\mathbb{R} \subseteq L)$ -equivalences

Theorem (Sierpinski 1965)

CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i -axis meets A_i in finitely many points.

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$\mathbb{R} \subseteq L$ holds iff there are Σ_2^1 sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i -axis meets A_i in finitely many points.

From CH-equivalences to $(\mathbb{R} \subseteq L)$ -equivalences

Theorem (Komjáth and Totik 2006)

CH holds iff there exists $g : \mathbb{R} \times \mathbb{R} \rightarrow \omega$ such that there are no two sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \upharpoonright C \times D$ is monochromatic.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there exists a Σ_2^1 -definable function $g : \mathbb{R} \times \mathbb{R} \rightarrow \omega$ such that there are no sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \upharpoonright C \times D$ is monochromatic.

From CH-equivalences to $(\mathbb{R} \subseteq L)$ -equivalences

Theorem (Komjáth and Totik 2006)

CH holds iff there exists a coloring $g : \mathbb{R} \rightarrow \omega$ such that there are no four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there exists a Σ_2^1 coloring $g : \mathbb{R} \rightarrow \omega$ such that there are no four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

From CH-equivalences to $(\mathbb{R} \subseteq L)$ -equivalences

Theorem (Erdős and Komjáth 1990)

CH holds if and only if the plane can be colored with countably many colors with no monochromatic right-angled triangle.

Theorem

$\mathbb{R} \subseteq L$ if and only if there exists a Σ_2^1 coloring of the plane with countably many colors with no monochromatic right-angled triangle.

CH-systems

$\langle P, (m_i)_{i \in \omega} \rangle$ is a CH-system if

- ▶ $P \subseteq \omega \times \omega_1 \times \mathbb{R}$;
- ▶ $\forall i \forall S \in [\omega_1]^{m_i} (\bigcap \{P_{i\alpha} \mid \alpha \in S\} \text{ is countable})$;
- ▶ $\{a : \forall i (|P_{ia}| < m_i)\}$ is countable.

Picture on the blackboard.

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Theorem

If there exists a CH-system then CH holds.

CH-systems

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- ▶ $\{a : \forall i (|P_{ia}| < m_i)\}$ is countable.

Fact

CH implies that there exists a CH-system.

Take an injection $f : \omega_1 \rightarrow \mathbb{R}$. For any i put

- ▶ $m_i = 1$,
- ▶ $P(i, \alpha, y) \equiv f(\alpha) \succ y$.

From CH-equivalences to CH-systems

Theorem (Sierpinski 1965)

CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Take an injection $f : \omega_1 \rightarrow \mathbb{R}$. For any i put

- ▶ $m_i = 1$,
- ▶ $P(i, \alpha, y) \equiv (f(\alpha), y) \in A$.

From CH-equivalences to CH-systems

Theorem (Sierpinski 1965)

CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i -axis meets A_i in finitely many points.

Take an injection $f : \omega_1 \rightarrow \mathbb{R}$. For any i put

- ▶ $m_i = 1$,
- ▶ $P(i, \alpha, y) \equiv \exists n((f(\alpha), y, n) \in A_2)$.

From CH-equivalences to CH-systems

Theorem (Komjáth and Totik 2006)

CH holds iff there exists $g : \mathbb{R} \times \mathbb{R} \rightarrow \omega$ such that there are no two sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \upharpoonright C \times D$ is monochromatic.

Take an injection $f : \omega_1 \rightarrow \mathbb{R}$. For any i put

- ▶ $m_i = 2$,
- ▶ $P(i, \alpha, y) \equiv g(f(\alpha), y) = i$.

From CH-equivalences to CH-systems

Theorem (Komjáth and Totik 2006)

CH holds iff there exists a coloring $g : \mathbb{R} \rightarrow \omega$ such that there are no four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

Take an injection $f : \omega_1 \rightarrow \mathbb{R}$. For any i put

- ▶ $m_i = 2$,
- ▶ $P(i, \alpha, y) \equiv g(f(\alpha) + y) = i$.

From CH-equivalences to CH-systems

Theorem (Erdős and Komjáth 1990)

CH holds if and only if the plane can be colored with countably many colors with no monochromatic right-angled triangle.

Take an injection $f : \omega_1 \rightarrow \mathbb{R}$. And $g : \mathbb{R} \rightarrow \omega$ a coloring which avoids right-angled triangles. For any i put

- ▶ $m_i = 2$,
- ▶ $P(i, \alpha, y) \equiv g(f(\alpha), y) = i$.

Σ_2^1 -CH-systems

P is a Σ_2^1 -CH-system if it is a CH-system and P is Σ_2^1 .

Theorem (Addison 1959)

If $\mathbb{R} \subseteq L$ then there exists a Δ_2^1 strong well-ordering of the reals.

Theorem (Mansfield and Solovay 1970)

Let A be a $\Sigma_2^1(a)$ set. Then either $A \subseteq L[a]$, or else A contains a perfect set. In particular, if a Σ_2^1 set contains a non-constructible real then it is uncountable.

Theorem

There exists a Σ_2^1 -CH-system if and only if $\mathbb{R} \subseteq L$ holds.

J-systems

$\langle P, (m_i)_{i \in \omega} \rangle$ is a CH-system if

- ▶ $P \subseteq \omega \times \omega_1 \times \mathbb{R}$;
- ▶ $\forall i \forall S \in [\omega_1]^{m_i} (\bigcap \{P_{i\alpha} \mid \alpha \in S\} \text{ is countable})$;
- ▶ $\{a : \forall i (|P_{ia}| < m_i)\}$ is countable.

$$J = \{X \subseteq \mathbb{R} : |X| \leq \omega\}$$

J-systems

$\langle P, (m_i)_{i \in \omega} \rangle$ is a *J-system* if

- ▶ $P \subseteq \omega \times \omega_1 \times \mathbb{R}$;
- ▶ $\forall i \forall S \in [\omega_1]^{m_i} (\bigcap \{P_{i\alpha} \mid \alpha \in S\} \in J)$;
- ▶ $\{a : \forall i (|P_{ia}| < m_i)\} \in J$.

Let J be a σ -ideal on \mathbb{R} (e.g. \mathcal{M}, \mathcal{N})

Theorem

There exists a J -system iff $\text{cov}(J) = \omega_1$.

Summing up

What other we have:

- ▶ The notion of (κ, κ^+) -system which is equivalent to $2^\kappa = \kappa^+$ for any infinite κ .
- ▶ The notion of \diamond -system which is equivalent to \diamond , for $P \subseteq \omega \times \omega_1 \times \mathcal{P}(\omega_1)$.

What we would like to have:

- ▶ A notion of \diamond -system for $P \subseteq \omega \times \omega_1 \times \mathcal{P}(\omega)$.
- ▶ Is it consistent to have a Π_1^1 -CH-system?
- ▶ Is there a $T \subseteq ZFC$ such that if

$$T \vdash \text{CH implies } \exists X \Phi(X),$$

then

$$T \vdash \text{CH implies } \exists X \in \Delta_1^1(\prec) \Phi(X)?$$

Summing up

What other we have:

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Thank you!