On Σ_1-Fixed Point Statements in KP

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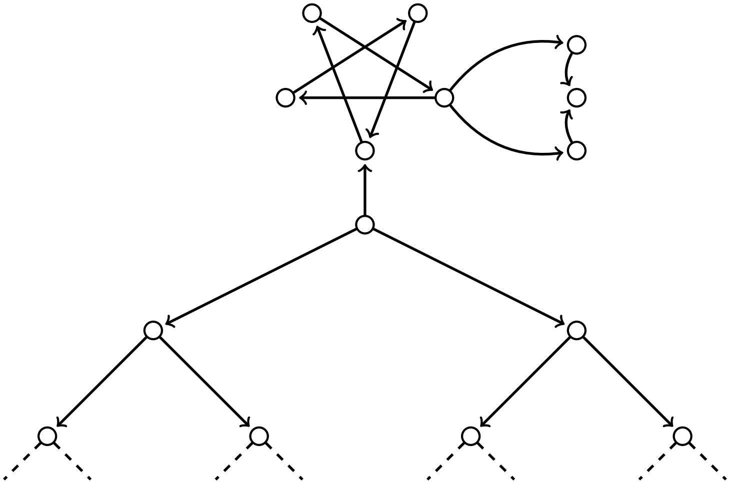
Let us start from a binary relation r



Well-founded part of a binary relation

The well-founded part of any binary relation $r \subseteq a \times a$ is the set of all $x \in a$ such that there are no infinite decreasing sequences from x.

So, which is the well-founded part of our relation?

...on the blackboard.

Well-founded part of a binary relation

It is known that we can obtain the well-founded part of any relation $r \subseteq a \times a$ has the least fixed point of the function

$$F(u) = \{x \in a : \forall z \in a(z \; r \; x \implies z \in u)\}.$$

This function is monotone, i.e.

$$u \subseteq v \implies F(u) \subseteq F(v).$$

Well-founded part of a binary relation

If there exists a least fixed point for this function, then it is the well-founded part of our relation.

Is leastness important?

...on the blackboard.

Does the least fixed point exist?

Kripke Platek Set Theory

We work in extensions of Kripke Platek Set Theory (KP). We briefly resume the axioms of KP.

- extensionality, pair, union, foundation, infinity,
- Δ_0-Separation: i.e, for every Δ_0 formula φ in which x is not free and any set a,
$$\exists x(x = \{y \in a : \varphi[y]\})$$
- Δ_0-Collection: i.e, for every Δ_0 formula φ and any set a,
$$\forall x \in a \exists y \varphi[x, y] \implies \exists b \forall x \in a \exists y \in b \varphi[x, y]$$

A first question

Given a set a and any monotone function $F : \mathcal{P}(a) \to \mathcal{P}(a)$, does there exist a set which is the least fixed point of F?

We extend the standard language with Σ_1-function symbols. F is a Σ_1-function symbol if there exists a Σ_1 formula φ such that:

- $\forall x \exists ! y (\varphi[x, y])$ (i.e, functional);
- $\forall x, y (F(x) = y \iff \varphi[x, y])$.

Σ_1-least fixed point

Σ_1-LFP

Given any set a and any Σ_1-function symbol F such that

1. $\forall x(F(x) \subseteq a)$ (i.e, bounded),
2. $\forall x, x'(x \subseteq x' \implies F(x) \subseteq F(x'))$ (i.e, monotone),

there exists z such that

- $F(z) = z$ (i.e, fixed point),
- $\forall x(F(x) = x \implies x \supseteq z)$ (i.e, leastness).

Σ_1-fixed point

Σ_1-FP

Given any set a and any Σ_1-function symbol F such that

1. $\forall x(F(x) \subseteq a)$ (i.e, bounded),
2. $\forall x, x'(x \subseteq x' \implies F(x) \subseteq F(x'))$ (i.e, monotone),

there exists z such that

- $F(z) = z$ (i.e, fixed point).

Σ_1-separation

Σ_1-separation

For every Σ_1 formula φ in which x is not free and any set a,

$$\exists x(x = \{y \in a : \varphi[y]\}).$$

Σ_1-separation implies Σ_1-LFP

- Given any set a and any F as in Σ_1-LFP, define by Σ-recursion:

$$I_\alpha = F(\bigcup\{I_\beta : \beta < \alpha\}).$$

- Define by Σ_1-Separation, the set

$$z = \{x \in a : \exists\alpha(x \in I_\alpha)\}.$$

- Σ-Reflection and monotonicity yield $z = I_\gamma$ for some ordinal γ.
- z is a set and it is the least fixed point.

Σ_1-separation implies Σ_1-LFP

Does the viceversa hold?

Σ_1-bounded proper injections

Σ_1-BPI

Given any set a and any Σ_1-function symbol F such that

- $\forall x(F(x) \in a)$,

there exist x and y such that

$$x \neq y \wedge F(x) = F(y).$$

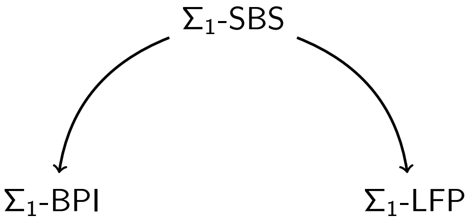
Σ_1-Subset Bounded Separation

Σ_1-SBS

For every Δ-formula φ and sets a and b,

$$\{x \in a : \exists y \subseteq b(\varphi[x, y])\}$$

is a set.



Σ_1-SBS implies Σ_1-BPI

- Given F and a as in Σ_1-BPI define by Σ_1-SBS the set

$$X = \{x \in a : \exists z \subseteq a(F(z) = x)\}.$$

- Suppose by contradiction that

$$\forall y, z \subseteq a(F(y) \neq F(z)).$$

- Define $h : X \to V$ such that

$$h(x) = \text{the unique } z \subseteq a(F(z) = x).$$

- We can prove that $\forall z(z \subseteq a \iff z \in h[X])$.
- We can conclude with the usual Cantor's argument.

Σ_1-SBS implies Σ_1-LFP

- Given F and a as in Σ_1-LFP, define

$$\mathrm{Cl}_F[y] \iff F(y) \subseteq y.$$

- By Σ_1-SBS we can define

$$z = \{x \in a : \forall y \subseteq a(\mathrm{Cl}_F[y] \implies x \in y)\}.$$

- We can prove that $F(z) = z$.
- Since every fixed point is closed under F, we have leastness.

Σ_1-Maximal Iteration

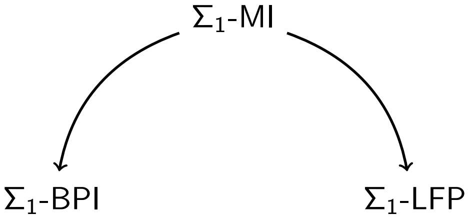
Σ_1-MI

Let a be any set and F be any Σ_1-function symbol such that

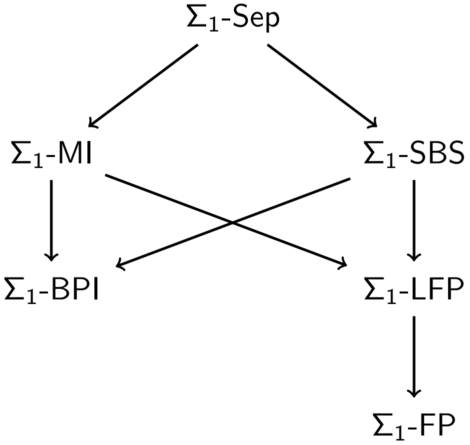
- $\forall x(F(x) \subseteq a)$ (i.e, bounded).

Then there exists α and f such that

- $\mathrm{fun}(f) \wedge \mathrm{dom}(f) = \alpha + 1$;
- $\forall \beta \leq \alpha(F(\bigcup_{\gamma \in \beta} f(\gamma)) = f(\beta))$
- $\bigcup_{\gamma \in \alpha} f(\gamma) \supseteq f(\alpha)$



Σ_1-fixed point principles in KP



Working with the Axiom of Constructibility (V=L)

Gödel's constructible universe is defined as follows:

- $L_0 = \emptyset$,
- $L_{\alpha+1} = \mathcal{P}(L_\alpha) \cap \mathcal{C}(L_\alpha \cup \{L_\alpha\})$,
- $L_\alpha = \bigcup \{L_\beta : \beta < \alpha\}$ for α limit,
- $L = \bigcup \{L_\alpha : \alpha \in \mathrm{On}\}$.

Where $C(x)$ is the closure under the Gödel's functions $\mathcal{F}_1, \ldots, \mathcal{F}_8$ of x.

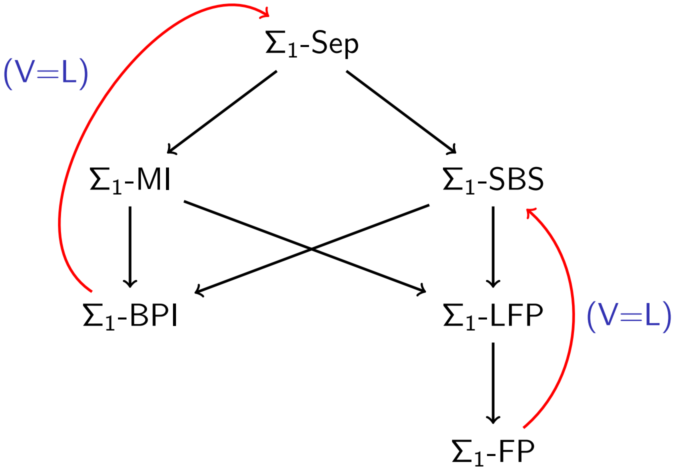
Working with the Axiom of Constructibility (V=L)

In KP + (V=L) the following implications hold:

- Σ_1-BPI implies Σ_1-Separation.
- Σ_1-FP implies Σ_1-SBS.

We can conclude that all our principles are not provable in KP + (V=L) since all of them are equivalent to Σ_1-Separation in this setting.

Σ_1-fixed point principles in KP + (V=L)



Are all these statements equivalent also in KP?

Beta

For any well-founded relation r on some set a there exists a function f such that:

$$\forall x \in a(f(x) = \{f(y) : (y, x) \in r\})$$

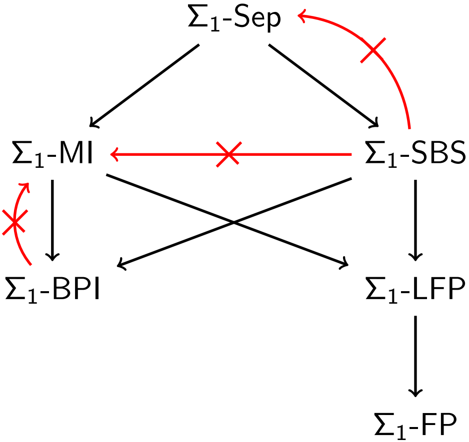
Are all these statements equivalent also in KP?

- Mathias proved that KP + Pow does not imply Axiom Beta

And we have

- Pow implies Σ_1-SBS,
- Σ_1-MI implies Beta.

Σ_1-fixed point principles in KP



Thank you!