Terminating via Ramsey's Theorem

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A first informal question

Assume that a child really likes biscuits, he has z-many biscuits. Assume that his grandmother gave him x-many gold coins and y-many silver coins to buy biscuits by pursuing the following rules at each purchase:

- the child may spend one silver coin to duplicate his number of biscuits;
- the child may spend one gold coin and all his silver coins to duplicate his number of biscuits and to get one silver coin for every biscuit he has.

Does the child get infinitely many biscuits?

A first formal question

```
while ( x > 0 AND y > 0 )
        ( x, y, z ) = ( x, y-1, 2*z )
      OR
        ( x, y, z ) = ( x-1, 2*z, 2*z )
```

Does this program terminate for any x, y and z?

Transition-based programs

A transition-based program $P = (S, I, R)$ consists of:

- S: a set of states,
- I: a set of initial states, such that $I \subseteq S$,
- R: a transition relation, such that $R \subseteq S \times S$.

A computation is a maximal sequence of states $s_0, s_2, \ldots$ such that

- $s_0 \in I$,
- $(s_{i+1}, s_i) \in R$ for any $i \in \mathbb{N}$.

$$s_i \longrightarrow s_{i+1}$$
$$s_{i+1} R s_i$$

The set Acc of accessible states is the set of all states which appear in some computation.

Termination Theorem by Podelski and Rybalchenko

- A program P is terminating if its transition relation R restricted to the accessible states is well-founded.
- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$.
- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem (Podelski and Rybalchenko 2004)

The program P is terminating if and only if there exists a disjunctively well-founded transition invariant for P.

Termination Theorem by Podelski and Rybalchenko

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- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem (Podelski and Rybalchenko 2004)

R is well-founded if and only if there exist $k \in \mathbb{N}$ and k-many well-founded relations $R_0, \ldots, R_{k-1}$ such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$.

An answer

```
while ( x > 0 AND y > 0 )
          ( x, y, z ) = ( x, y-1, 2*z )
       OR
          ( x, y, z ) = ( x-1, 2*z, 2*z )
```

A transition invariant for this program is $R_1 \cup R_2$, where

$$R_1 := \{(\langle x', y', z' \rangle, \langle x, y, z \rangle) \mid y > 0 \wedge y' < y\}$$
$$R_2 := \{(\langle x', y', z' \rangle, \langle x, y, z \rangle) \mid x > 0 \wedge x' < x\}$$

Since each R_i is well-founded, then the program terminates.

A second question

```
while ( x > 0 AND y > 0 )
          ( x, y, z ) = ( x, y-1, 2*z )
      OR
          ( x, y, z ) = ( x-1, 2*z, 2*z )
```

How many steps before the program terminates?

I.e. how many biscuits can the child get?

Infinite Ramsey Theorem for pairs

If you have $\mathbb{N}$-many people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem (Ramsey 1930)

For any $k \in \mathbb{N}$ and for every k-coloring $c : [\mathbb{N}]^2 \to k$, there exists an infinite homogeneous set H, i.e. there exists $h < k$, such that for any distinct $x, y \in H$, $c(\{x, y\}) = h$.

Complete disorder is impossible

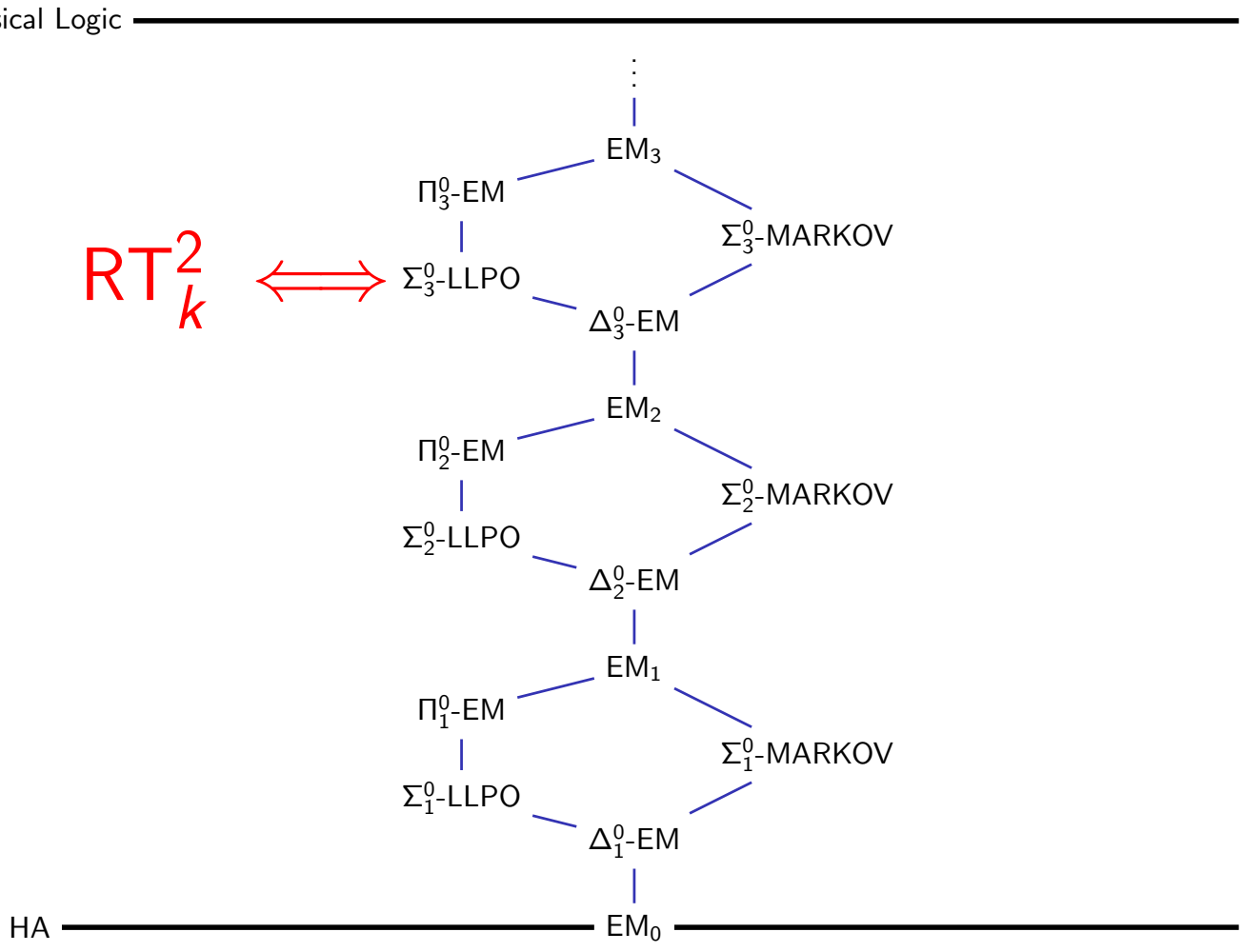
Theodore Samuel Motzkin

How many steps before the program terminates?

Hard to say, because Ramsey's Theorem is a purely classical result. Indeed,

- In 1969 Specker proved there is one recursive coloring in two colors with no recursive infinite homogeneous sets.
- In 1972 Jockusch proved that it is not even possible to recursively find a color for which there is an infinite homogeneous set.

Ramsey's Theorem in the hierarchy of classical principles



H-closure Theorem

A binary relation R is H-well-founded there are no infinite decreasing transitive R-sequences.

Theorem (Berardi and S. 2014)

For any $k \in \mathbb{N}$, if $R_0, \ldots, R_{k-1}$ are H-well-founded relations, then $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded.

- H-closure Theorem is classically true, because there exists a simple (i.e. within RCA_0) classical proof of the equivalence between Ramsey's Theorem and H-closure Theorem.
- By considering the inductive definition of well-foundedness, this result is intuitionistically provable.

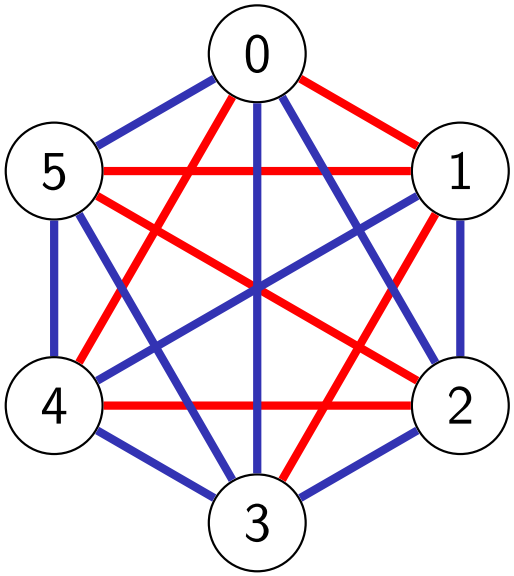
Well-foundedness

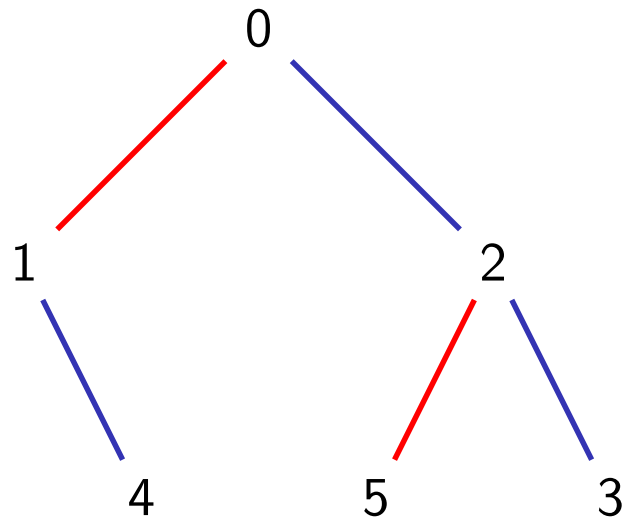
Let R be a binary relation on S:

- R is weakly well-founded if there are no infinite R-decreasing sequences.
- R is classically well-founded if every R-decreasing sequence is finite.
- R is inductively well-founded if every element in S belongs to every R-inductive set.
- R is strongly well-founded if every inhabited subset of S has a R-minimal element.

Erdős' trees

Assume given a sequence $\langle 0, \ldots, 5 \rangle$ such that the coloring between its elements is as follows.

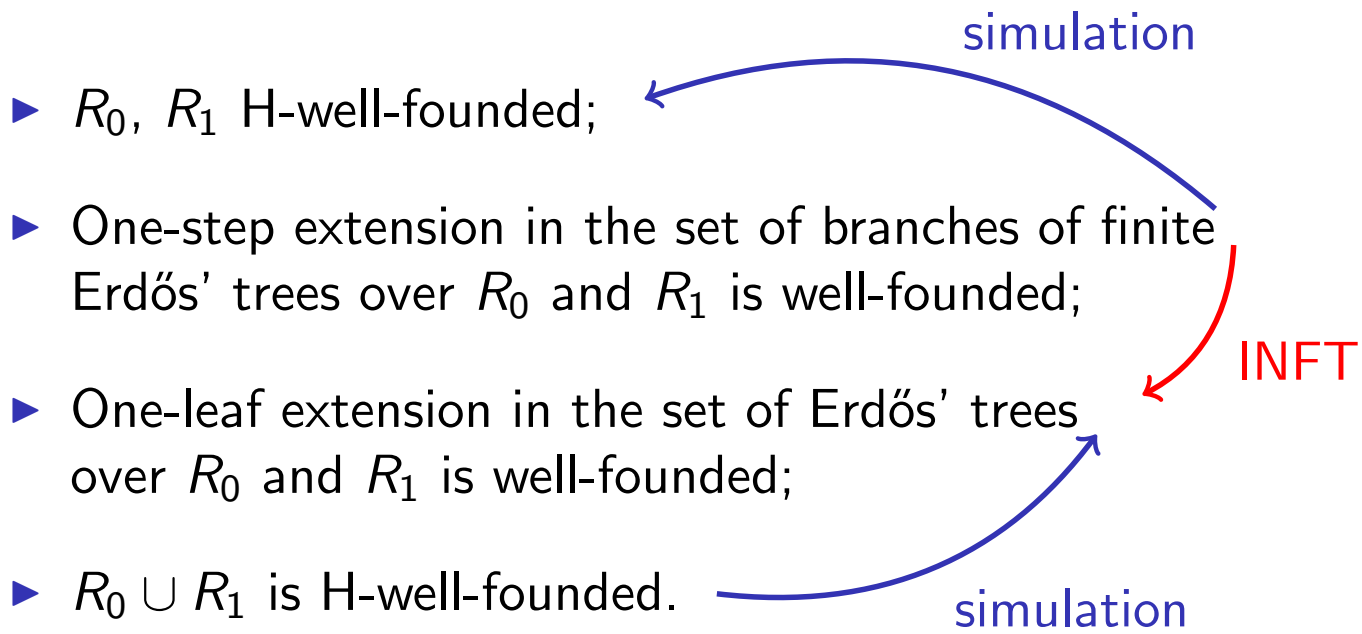




If $x \prec_E y \prec_E z$, then $c(\{x, y\}) = c(\{x, z\})$.

An intuitionistic proof of the H-closure Theorem

T is a simulation of R in Q if for every x', x, y,
$(x'Rx \wedge xTy) \implies \exists y'(y'Qy \wedge x'Ty')$.



- R_0, R_1 H-well-founded;
- One-step extension in the set of branches of finite Erdős' trees over R_0 and R_1 is well-founded;
- One-leaf extension in the set of Erdős' trees over R_0 and R_1 is well-founded;
- $R_0 \cup R_1$ is H-well-founded.

Intuitionistic Nested Fan Theorem (INFT): if one-step extension in the set of branches of a set of trees is well-founded, then one-leaf extension in such a set of trees is well-founded.

An intuitionistic proof of the Termination Theorem

Assume that there exists a disjunctively well-founded transition invariant, namely

$$R_0 \cup \cdots \cup R_{k-1} \supseteq R^+,$$

- then R_i is H-well-founded for each $i < k$;
- hence $R_0 \cup \cdots \cup R_{k-1}$ is H-well-founded;

H-closure

- therefore R^+ is H-well-founded and transitive;
- so it is well-founded, and then also R is.

Bounds from H-closure Theorem

A weight function for a binary relation $R \subseteq S^2$ is a function $f : S \to \mathbb{N}$ such that for any $x,\ y \in S$

$$xRy \implies f(x) < f(y).$$

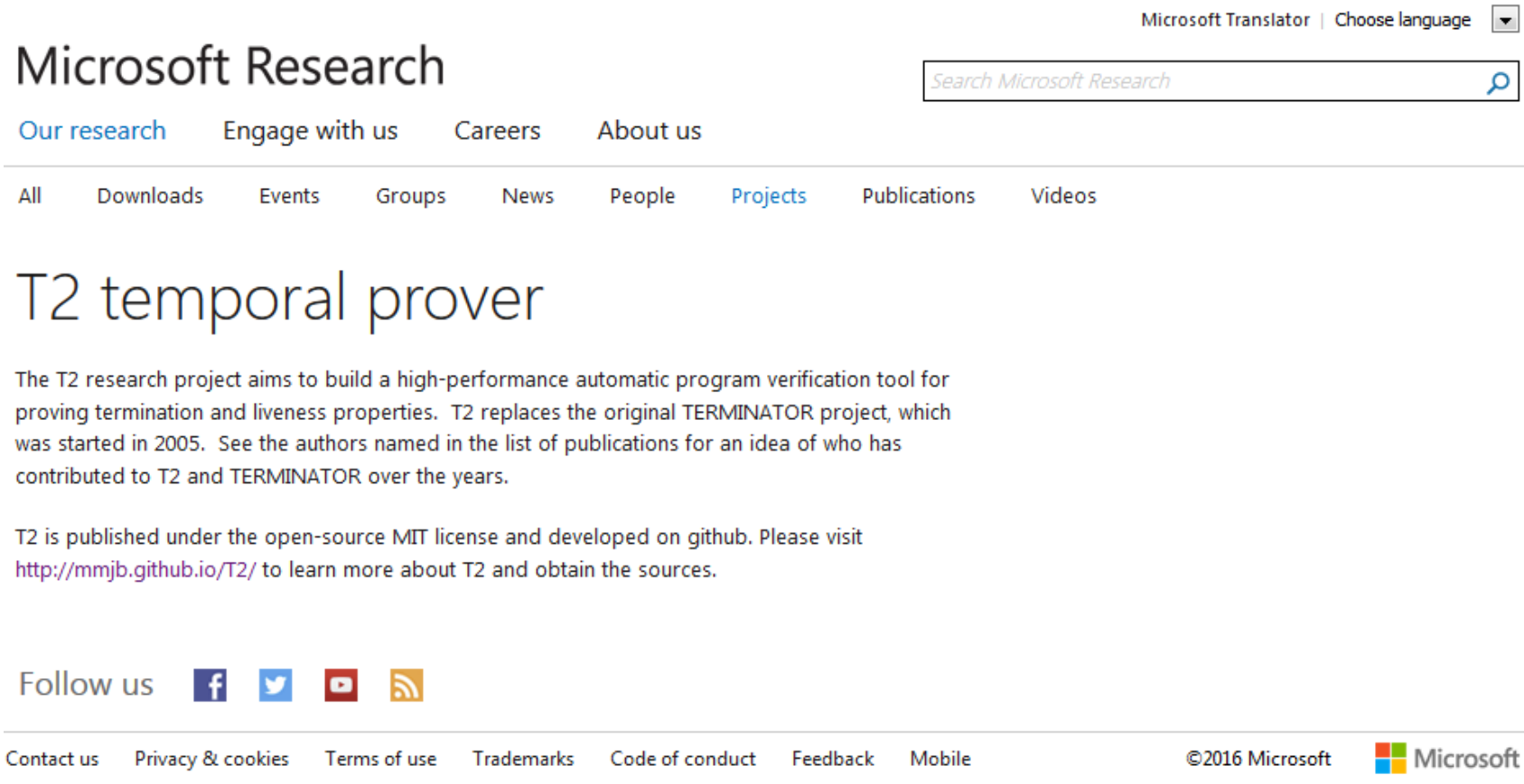
$\mathcal{A}$ = the class of functions computable by a program for which there exists a disjunctively well-founded transition invariant whose relations have primitive recursive weight functions.

Proposition (Berardi, Oliva and S. 2014)

$$\mathcal{A} = \mathrm{PR}$$

Computation → Erdős' Tree → Height → Bound

And for Terminator



May we consider the classical definition of well-foundedness?

- Gödel's system T is simply typed λ-calculus enriched with natural numbers and primitive recursion in all finite types, together with the associated reduction rules.

- Spector's bar recursion can be intuitively explained as a recursive definition of a function through the set of the nodes of a well-founded tree.

- The Dialectica interpretation of arithmetic was extended by Spector to classical analysis in the system "T + bar recursion".

A bar recursive bound

The function $\mu\colon {}^{\mathbb{N}}S \to \mathbb{N}$ is a modulus of well-foundedness for R if

$$\forall \sigma\ \exists i < \mu(\sigma)\ \neg(\sigma_{i+1} R \sigma_i)$$

Theorem (Berardi, Oliva and S. 2014)

There exists a construction Φ, definable in T+ bar recursion, such that for all $k \in \mathbb{N}$ and $R, R_0, \ldots, R_{k-1}$ such that

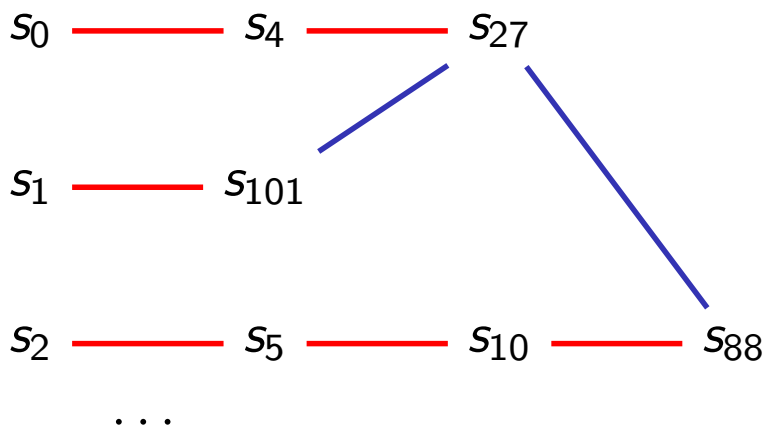
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$
- there exists μ_i modulus of well-foundedness for R_i,

$\Phi(R, \mu_0, \ldots, \mu_{k-1}, R_0, \ldots, R_{k-1})$ is a modulus of well-foundedness for R.

Due to a result by Schwichtenberg, if $\mu_0, \ldots, \mu_{k-1}$ are in system T, then also Φ is. In the case $\mu_0, \ldots, \mu_{k-1}$ are in system T_0, we only know that Φ is in T.

A bar recursive bound

- Assume that μ_0 and μ_1 are moduli of well-foundedness for R_0 and R_1. Let $s_0, s_1, \ldots$ be a computation, i.e. $\forall i(s_{i+1}(R_0 \cup R_1)s_i)$.



- μ_0, μ_1 provide that this sequence of sequences is finite. The greatest element is the bound.
- The construction is primitive recursive on some oracle γ which provides the next element connected in color red.
- Bar recursion yields an approximation of γ large enough to conclude that s is finite.

Might a Reverse Mathematical approach help?

- Which bounds may we get by using Reverse Mathematical tools?
- (Gasarch) Is there a natural example showing that the Termination Theorem requires the full Ramsey Theorem for pairs?
- (Gasarch) Is the Termination Theorem equivalent to Ramsey's Theorem for pairs?

The Termination Theorem in the Ramsey's zoo

- WRT^2_k. For any $c : [\mathbb{N}]^2 \to k$, there exists an infinite weakly homogeneous set; i.e. there exist $h \in k$ and $H = \{x_i : i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for any $i \in \mathbb{N}$ $c(x_i, x_{i+1}) = h$.

- CAC. Every infinite poset has an infinite chain or antichain.

- k-TT. For any relation R, if there exist $R_0, \ldots, R_{k-1}$ such that they are well-founded and $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$, then R is well-founded.

Theorem (S. and Yokoyama 2015)

$$\mathrm{RCA}_0 \vdash \forall k(k\text{-}\mathrm{TT} \iff \mathrm{WRT}_k).$$

Answers to questions posed by Gasarch

Theorem (Hirschfeldt and Shore 2007)

CAC plus full induction does not imply RT^2_2.

Since CAC plus full induction proves $\forall k$ k-TT:

- Is there a natural example showing that the Termination Theorem requires the full Ramsey Theorem for pairs? NO!
- Is the Termination Theorem equivalent to Ramsey's Theorem for pairs? NO!

Hence, which bounds may we get by using Reverse Mathematical tools?

Bounds and H-bounds

Let R be a binary relation on S.

- A weight function for R is a function $f : S \to \mathbb{N}$ such that for any $x,\ y \in S$
$$xRy \implies f(x) < f(y).$$

- A bound for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing sequence $a_{l-1}R \ldots Ra_0$, $l \leq f(a_0)$.

- A H-bound for R is a function $f : S \to \mathbb{N}$ such that for any R-decreasing transitive sequence $a_{l-1}R \ldots Ra_0$, $l \leq f(a_0)$.

Reverse mathematical bounds

Theorem (Parson 1970 / Paris and Kirby 1977 / Chong, Slaman and Yang 2012)

The class of provable recursive functions of $\mathrm{WKL}_0 + \mathrm{CAC}$ is exactly the same as the class of primitive recursive functions.

Consequence

Any relation R generated by a primitive recursive transition function for which there exist k-many relations $R_0, \ldots, R_{k-1}$ with primitive recursive weight functions such that $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$ has a primitive recursive bound.

Another question

Summing up, if we assume that R, $R_0, \ldots, R_{k-1}$ are such that:

- R is the graph of a primitive recursive function;
- R_i has a primitive recursive weight function;
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$;

Then R has a primitive recursive bound.

Is there a correspondence between the complexity of the bound and

- k?
- the complexity of the weight functions for $R_0, \ldots, R_{k-1}$?

Paris-Harrington Theorem for pairs

For given $k \in \mathbb{N}$,

- PH^{*2}_k: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a homogeneous set H for c such that $\min H < |H|$.

- WPH^{*2}_k: for any infinite set $X \subseteq \mathbb{N}$ and any coloring function $c : [X]^2 \to k$, there exists a weakly homogeneous set H for c such that $\min H < |H|$.

Fast Growing Hierarchy

Let F_k be the usual k-th fast growing function defined as

$$\begin{cases} F_0(x) = x + 1, \\ F_{h+1}(x) = F_h{}^{(x+1)}(x). \end{cases}$$

Let $\mathrm{Tot}(F_k)$ denote the totality of F_k:

$$\forall a \exists b (F_k(a) = b).$$

Let k-TT^h for weight functions (respectively bounds or H-bounds) be the following statement:

Assume that R, $R_0, \ldots, R_{k-1}$ are such that:

- R is the graph of a function below F_h;
- R_i has a weight function (respectively bound or H-bound) below F_h;
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$.

Then R has a bound.

With Keita Yokoyama, we proved that there is an equivalence over RCA_0^*

- between k-TT^h for weight functions, k-TT^h for bounds and some restricted version of WPH^{*2}_k ($\mathrm{WPH}^{h,2}_k$).
- k-TT^h for H-bounds and some restricted version of PH^{*2}_k ($\mathrm{PH}^{h,2}_k$).

From transition invariants to bound

Theorem (Solovay and Ketonen 1981)

In RCA_0^*. $\mathrm{Tot}(F_{k+h+5}) \implies \mathrm{PH}_k^{h,2}$.

Consequence

For any $R, R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that

- R is the graph of a function below F_h,
- R_i has a H-bound below F_h;
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$.

R is bounded by F_{k+h+5}.

Is it possible to improve it?

In 2011 Figueira D., Figueira S, Schmitz and Schnoebelen observed that the Termination Theorem is a consequence of Dickson's Lemma

Theorem (Dickson 1913)

For any natural number k, every infinite sequence σ of elements in $\mathbb{N}^k$ is good; i.e. for any infinite sequence σ of elements in $\mathbb{N}^k$ there exist natural numbers $n < m$ such that $\sigma(n) \leq \sigma(m)$.

Note that given a transition-based program $P = (S, I, R)$, for which there is k-disjunctively well-founded transition invariant composed of relations with weight functions we can define a map:

$$\begin{array}{rccc} \sigma: & S & \longrightarrow & \mathbb{N}^k \\ & s & \longmapsto & (f_0(s), \ldots, f_{k-1}(s)) \end{array}$$

where f_i is a weight function of R_i.

Any computation is mapped in a bad sequence!

$$s_i \longrightarrow s_{i+1}$$

$$s_{i+1} R_h s_i$$

$$f_h(s_{i+1}) < f_h(s_i)$$

$$\sigma(s_i) > \sigma(s_{i+1})$$

Bounding bad sequences

Figueira D., Figueira S., Schmitz and Schnoebelen, provided a bound for the length of the bad sequences. As a corollary

Theorem (Figueira D., Figueira S., Schmitz and Schnoebelen 2011)

For any $R, R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that

- R is the graph of a function below F_h;
- R_i has bound below F_h;
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$;

R is bounded by $F_{k+\max\{1,h-1\}}$.

Consequence

In RCA_0^*. $\mathrm{Tot}(F_{k+\max\{1,h-1\}}) \implies \mathrm{WPH}_k^{h,2}$.

And for H-bounds?

By looking for bounds via Erdős' tree.

Theorem (S. 2015)

In RCA_0^*. For any $R, R_0, \ldots, R_{k-1} \subseteq \mathbb{N}^2$ such that

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Consequence

In RCA_0^*. $\mathrm{Tot}(F_{k+\max\{h-1,1\}}) \implies \mathrm{PH}_k^{h,2}$.

Example of OPTIMAL bounds

```
while (x > 0 AND y > 0)
    if(y > 1)
                  (x,y,z) = (x, y-1, 2*z)
        else      (x,y,z) = (x-1,2*z, 2*z)
```

A transition invariant for this program is $R_1 \cup R_2$, where

$$R_1 := \{(\langle x', y', z' \rangle, \langle x, y, z \rangle) \mid y > 0 \wedge y' < y\} \qquad \text{Bounded by } F_0$$
$$R_2 := \{(\langle x', y', z' \rangle, \langle x, y, z \rangle) \mid x > 0 \wedge x' < x\} \qquad \text{Bounded by } F_0$$

Then R is well-founded, and R is bounded by F_{2+1}!

It is optimal since for any $x \geq y > 0$, the computation which starts in $(x, y, 1)$ has length greater than $F_2^{x-2}(y)$!

Comparison with H-closure argument

By analysing the proof of the bound obtained from the H-closure Theorem:

If R, $R_0, \ldots, R_{k-1}$ are such that:

- R is the graph of a function below F_h;
- R_i has a weight function below F_h;
- $R_0 \cup \cdots \cup R_{k-1} \supseteq R^+$;

Then R is has is bounded by $F_{k+\max\{1,h\}}$.

Size-Change Termination programs

A functional program is SCT if for every sequence of calls: the sequence is infinite implies there is some infinite descent (i.e. weakly decreasing sequence of values which is strictly decreasing infinitely many times).

A functional program is SCT* if for every sequence of calls: "there are no infinite descent" implies that the sequence is well-founded.

```
f(x,y,z) = if (y > 1)   c0: f(x,y-1, 2*z)
           else         c1: f(x-1,2*z,2*z).
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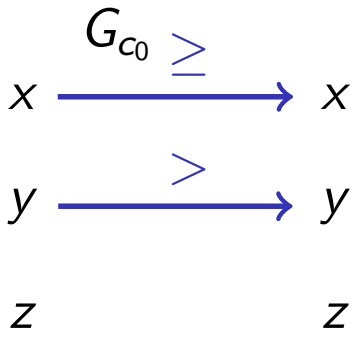
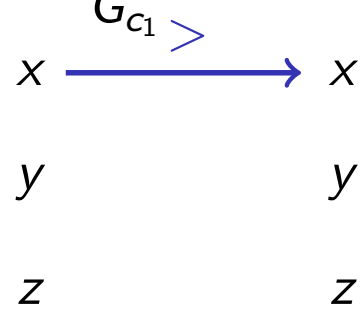
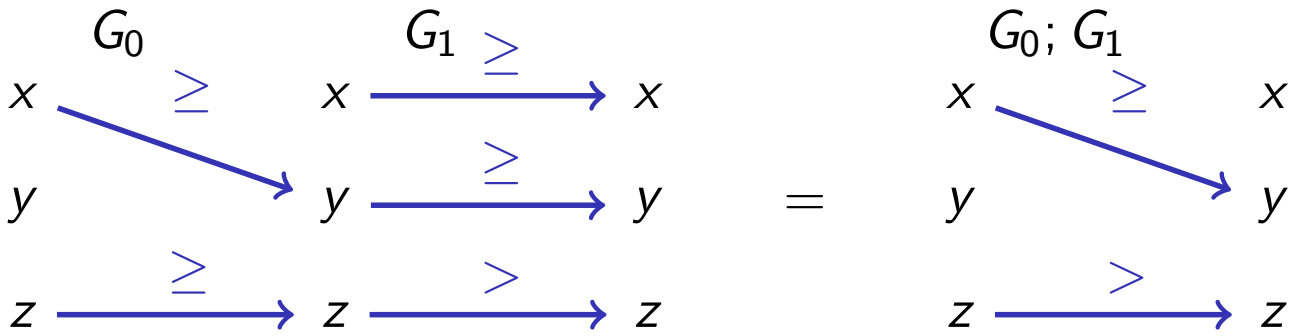




Size-Change Termination Theorem

A size-change graph G is idempotent if $G; G = G$, where



Theorem (Lee, Jones, Ben-Amram 2001)

A program is SCT if and only if every idempotent size-change graph of the program has an edge $x \xrightarrow{>} x$.

Size-Change Termination Theorem

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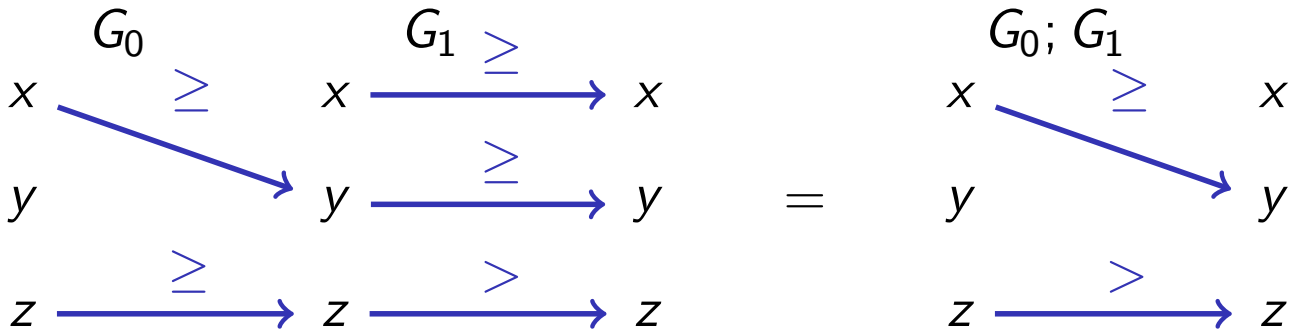

Theorem (S. 2015)

A program is SCT* if and only if every idempotent size-change graph of the program has an edge $x \xrightarrow{\geq} x$.

Bounds from H-closure Theorem

Consider tail-recursive functional programs.

If for any G idempotent $x \xrightarrow{<} x \in G$, then we can build a disjunctively well-founded transition invariant for its transition-based translation whose relations have primitive recursive weight functions.

This yields a different proof of:

Theorem (Ben-Amram 2002)

The functional programs which are tail-recursive and SCT computes exactly the primitive recursive functions.

Some further directions

1. Tighter bounds for SCT. Up to now, the transition invariant derived for tail-recursive SCT programs is quite large.

2. A reverse mathematical analysis of SCT. Since the Ackermann function is SCT, it seems that higher induction should be use to prove termination from SCT.

3. Extract more information from the bar-recursive construction of Φ, to compare this approach with the other ones.

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Thank you!