SCT through the reverse mathematical looking glasses

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A first informal question

Assume that there is a guy, who is hiking. He has lots of motivation (x) and some energy (y). To hike he has to pursue the following rules:

- the guy may spend one energy point to hike for 1 km / 200 v.m. ;
- the guy may spend one motivation point and all energy points to hike for 3 km / 600 v.m. and to get one energy point for every 100 v.m. he has done. (... since the view is amazing!)

Does the guy keep hiking forever?

A first formal question

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z)

            else if (y > 1)
                        then  c0: f( x, y-1, z + 200 )

                  else  c1: f( x-1, z : 100, z + 600 )
```

Does this program terminate for any x, y and z?

Programs

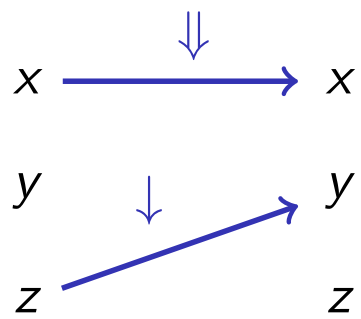
We consider a basic first order functional language:

$$
\begin{array}{rll}
x \in \mathrm{Par} & \text{parameter identifier} \\
f \in \mathrm{Fun} & \text{function identifier} \\
o \in \mathrm{Op} & \text{primitive operator} \\
a \in \mathrm{AExp} & \text{arithmetic expression} \\
& ::= x \mid x+1 \mid x-1 \mid o(a,\ldots,a) \mid f(a,\ldots,a) \\
b \in \mathrm{BExp} & \text{boolean expression} \\
& ::= x=0 \mid x=1 \mid x<y \mid x \leq y \mid b \wedge b \mid b \vee b \mid \neg b \\
e \in \mathrm{Exp} & \text{expression} \\
& ::= a \mid \textbf{if}\ \ b\ \ \textbf{then}\ \ e\ \ \textbf{else}\ \ e \\
d \in \mathrm{Def} & \text{function definition} \\
& ::= f(x_0,\ldots,x_{n-1}) = e \\
P \in \mathrm{Prog} & \text{program} \\
& ::= d_0,\ldots,d_{m-1}
\end{array}
$$

Size change Graphs

Let P be a program and $f,\ g \in \mathrm{Fun}(P)$.

- A size-change graph $G : f \to g$ for P is a bipartite graph on $(\mathrm{Par}(f), \mathrm{Par}(g))$.
- The set of edges is a subset of $\mathrm{Par}(f) \times \{\downarrow, \Downarrow\} \times \mathrm{Par}(g)$ such that there is at most one edge for any $x \in \mathrm{Par}(f)$ and $y \in \mathrm{Par}(g)$.

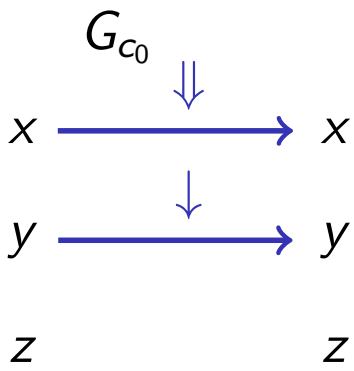


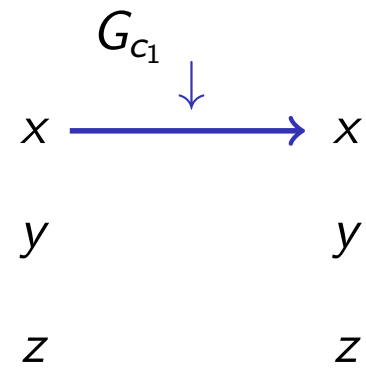
f is the source function of G and g is the target function of G.

Safe size change Graphs

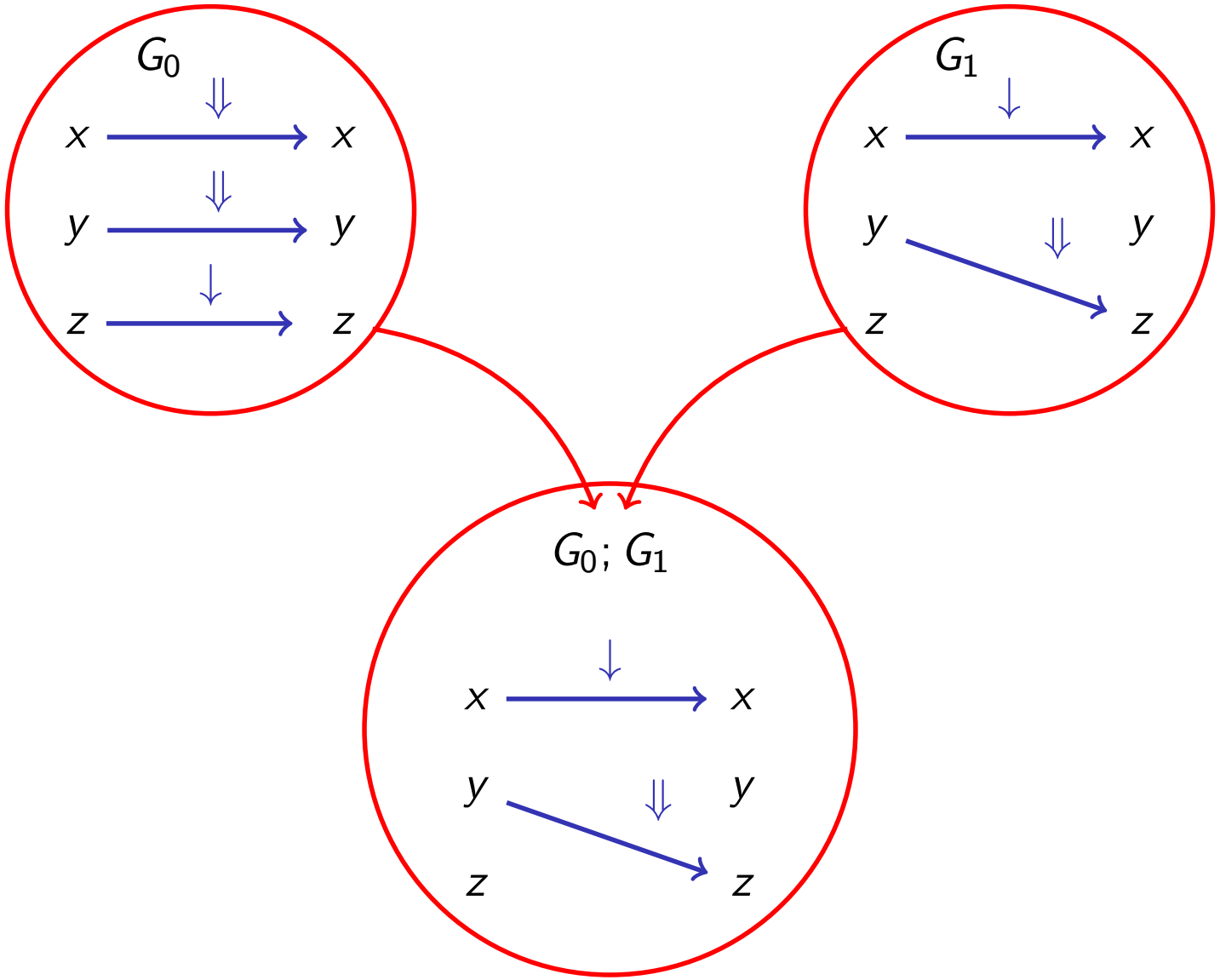
A size-change graph $G : f \to g$ for a call $\tau : f \to g$ is safe if it reflects the relationship between the parameter values in the program call.

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z)
                 else if (y > 1)  c0: f( x, y-1, z + 200 )
                      else  c1: f( x-1, z : 100, z + 600 )
```





Composition of size change graphs



Multipaths and Threads

- G is idempotent if $G; G = G$.
- Given a finite set of size-change graphs $\mathcal{G}$, $\mathsf{cl}(\mathcal{G})$ is the smallest set which contains $\mathcal{G}$ and is closed by composition.
- A multipath $\mathcal{M}$ is a sequence $G_0, \ldots, G_n, \ldots$ of graphs such that the target function of G_i is the source function of G_{i+1} for all i.
- A thread is a connected path of edges in $\mathcal{M}$ that starts at some G_t, where $t \in \mathbb{N}$.
- A multipath $\mathcal{M}$ has infinite descent if some thread in $\mathcal{M}$ contains infinitely many decreasing edges.

SCT Criterion

- A description $\mathcal{G}$ of P is a finite set of size-change graphs such that to every call $\tau : f \to g$ of P corresponds exactly one $G_\tau \in \mathcal{G}$.
- A description $\mathcal{G}$ of P is safe if each graph in $\mathcal{G}$ is safe.
- We say that a description $\mathcal{G}$ of P is size-change terminating (MSCT) if every infinite multipath $\mathcal{M} = G_0, \ldots, G_n, \ldots$, where every graph $G_n \in \mathcal{G}$, has an infinite descent.

Theorem (Lee, Jones, Ben Amram 2001)

A program P is MSCT iff every idempotent $G \in \mathsf{cl}(\mathcal{G}_P)$ has an arc $x \overset{\downarrow}{\to} x$.

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Theorem (Lee, Jones, Ben Amram 2001)

Let $\mathcal{G}$ be a description of P. Then $\mathcal{G}$ is SCT iff every idempotent $G \in \mathsf{cl}(\mathcal{G})$ has an arc $x \overset{\downarrow}{\to} x$.

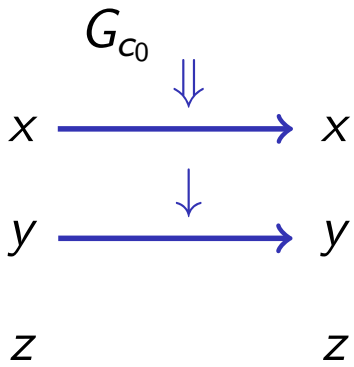
From SCT to termination

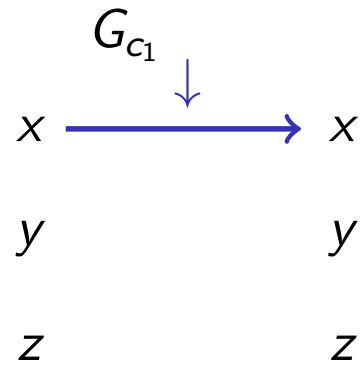
- A multipath $\mathcal{M}$ has infinite descent if some thread in $\mathcal{M}$ contains infinitely many decreasing edges.
- We say that a description $\mathcal{G}$ of P is MSCT if every infinite multipath $\mathcal{M} = G_0, \ldots, G_n, \ldots$, where every graph $G_n \in \mathcal{G}$, has an infinite descent.
- We say that a description $\mathcal{G}$ of P is ISCT if every idempotent $G \in \mathsf{cl}(\mathcal{G})$ has an arc $x \stackrel{\downarrow}{\rightarrow} x$.
- A program P with a safe MSCT description does not have infinite state transition sequences.
- Thus the existence of a safe ISCT description is a sufficient condition for termination.

An answer

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z)
            else if (y > 1)  c0: f( x, y-1, z + 200 )
                 else  c1: f( x-1, z : 100, z + 600 )
```

A safe description:





Since every idempotent $G \in \mathsf{cl}(\mathcal{G})$ has an arc $x \overset{\downarrow}{\to} x$, then the program terminates.

A second question

```
f(x,y,z) := if ( x = 0 OR y =  0 ) then (x,y,z)
            else if (y > 1)  c0: f( x, y-1, z + 200 )
                 else  c1: f( x-1, z : 100, z + 600 )
```

Where can we prove that this program terminates?

Reverse Mathematics

Given a theorem of ordinary mathematics, what is the weakest subsystem of second order arithmetic in which it is provable?

- RCA_0: axioms of arithmetic, Σ^0_1-induction, Δ^0_1-comprehension.
- WKL_0: RCA_0, Σ^0_1-separation.
- ACA_0: RCA_0, arithmetical comprehension.
- ATR_0: ACA_0, Σ^1_1-separation.
- Π^1_1-CA_0: ACA_0, Π^1_1-comprehension.

Γ-induction: for any $\varphi(x)$ in Γ,

$$(\varphi(0) \wedge \forall n(\varphi(n) \implies \varphi(S(n)))) \implies \forall n \varphi(n).$$

Γ-comprehension: for any $\varphi(x)$ in Γ,

$$\exists X \forall n(n \in X \iff \varphi(n)).$$

Γ-separation: for any $\psi(x)$, $\varphi(x)$ in Γ which are exclusive,

$$\exists X \forall n((\psi(n) \implies n \in X) \wedge (\varphi(n) \implies n \notin X)).$$

Where can we prove that this program terminates?

From Lee, Jones and Ben-Amram's paper we can outline the following three-steps argument to prove the termination of a first order functional program P:

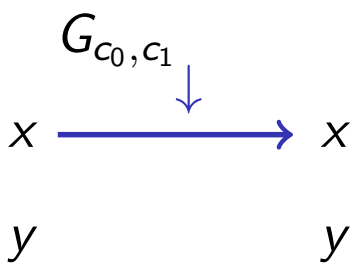
1. Verify that P is ISCT;
2. Apply the SCT criterion to prove that P is MSCT;
3. Derive the termination of P from the fact that "every MSCT program terminates"

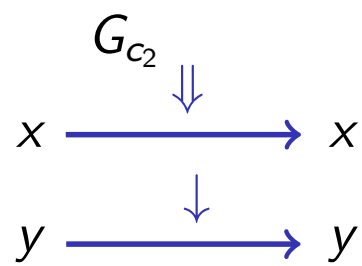
Questions:

- In which subsystem can we prove the SCT criterion?
- In which subsystem can we prove that every MSCT program terminates? (i.e. Soundness)

A safe description for Ackerman

```
A(x,y) := if ( x = 0) then y+1;
          else if (y = 0) then  c0: A(x-1,1);
          else  c1: A(x-1, c2: A(x,y-1))
```





Infinite Ramsey Theorem for pairs

If you have $\mathbb{N}$-many people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

(RT^1_k) For any $c\colon \mathbb{N} \to k$ there exists $i < k$ such that $c(x) = i$ for infinitely many x.

(RT^1) $\forall k \in \mathbb{N}\ \mathrm{RT}^1_k$.

(RT^2_k) For any $c : [\mathbb{N}]^2 \to k$ there exists an infinite homogeneous set $X \subseteq \mathbb{N}$, that is $c''[X]^2$ is constant.

(RT^2) $\forall k \in \mathbb{N}\ \mathrm{RT}^2_k$.

One direction of the SCT criterion is already provable in RCA_0

Proposition (RCA_0)

Let $\mathcal{G}$ be a finite set of size-change graphs. If every multipath $M = G_0, \ldots, G_n, \ldots$ has an infinite descent (MSCT), then every idempotent $G \in \mathrm{cl}(\mathcal{G})$ has an arc $x \overset{\downarrow}{\to} x$ (ISCT).

- Let G be idempotent.
- Then $M = G, G, \ldots, G, \ldots$ is a multipath.
- By hypothesis there exists an infinite descent.
- Define an infinite sequence $x_0, x_1, x_2, \ldots$ such that $x_i \overset{\downarrow}{\to} x_{i+1} \in G$.
- By the finite pigeonhole principle there exist $i < j$ such that $x = x_i = x_j$.
- By idempotence of G, $x \overset{\downarrow}{\to} x \in G$.

Where can we prove the other direction?

Proposition (Lee, Jones and Ben-Amram)

Let $\mathcal{G}$ be a finite set of size-change graphs. If every idempotent $G \in \mathrm{cl}(\mathcal{G})$ has an arc $x \overset{\downarrow}{\to} x$ (ISCT) then every multipath $M = G_0, \ldots, G_n, \ldots$ has an infinite descent (MSCT).

- Let $\mathcal{M}_\pi = G_0, \ldots, G_n, \ldots$.
- Define $c(i, j) = G_i; \ldots; G_{j-1}$.
- By RT^2 to c, there exists an infinite homogeneus set H.
- Let $i < j < k \in H$. $G = G_i; \ldots; G_{j-1}$ is idempotent, indeed $G; G = c(i, j); c(j, k) = c(i, k) = G$.
- By hypothesis, we have that there exists $x \overset{\downarrow}{\to} x \in G$.
- We prove that there exists an infinite descent starting from x in $G_{\min H}$.

Some corollaries of Ramsey's Theorem

(Triang_k) For any coloring $c : [\mathbb{N}]^2 \to k$ there exist $i \in k$ and $t \in \mathbb{N}$ such that $c(t, m) = c(t, l) = c(m, l) = i$ for infinitely many pairs $\{m, l\}$.

(Triang) $\forall k \in \mathbb{N}\ \mathrm{Triang}_k$.

(SPP_k) For any coloring $c : \mathbb{N} \to k$ there exists $I \subseteq k$ such that $i \in I$ iff $i < k$ and $c(x) = i$ for infinitely many x.

(SPP) $\forall k \in \mathbb{N}\ \mathrm{SPP}_k$.

Where can we prove the SCT criterion?

Theorem (Frittaion, S., Yokoyama 2017)

Over RCA_0:

- Triang implies the SCT criterion.
- SCT criterion implies SPP.
- $\mathsf{I}\Sigma^0_2$ implies Triang.

The SCT criterion is equivalent to $\mathsf{I}\Sigma^0_2$.

Where can we prove soundness?

Proposition (Lee, Jones and Ben-Amram)

Let $\mathcal{G}$ be a safe MSCT description of a program P, then P is terminating.

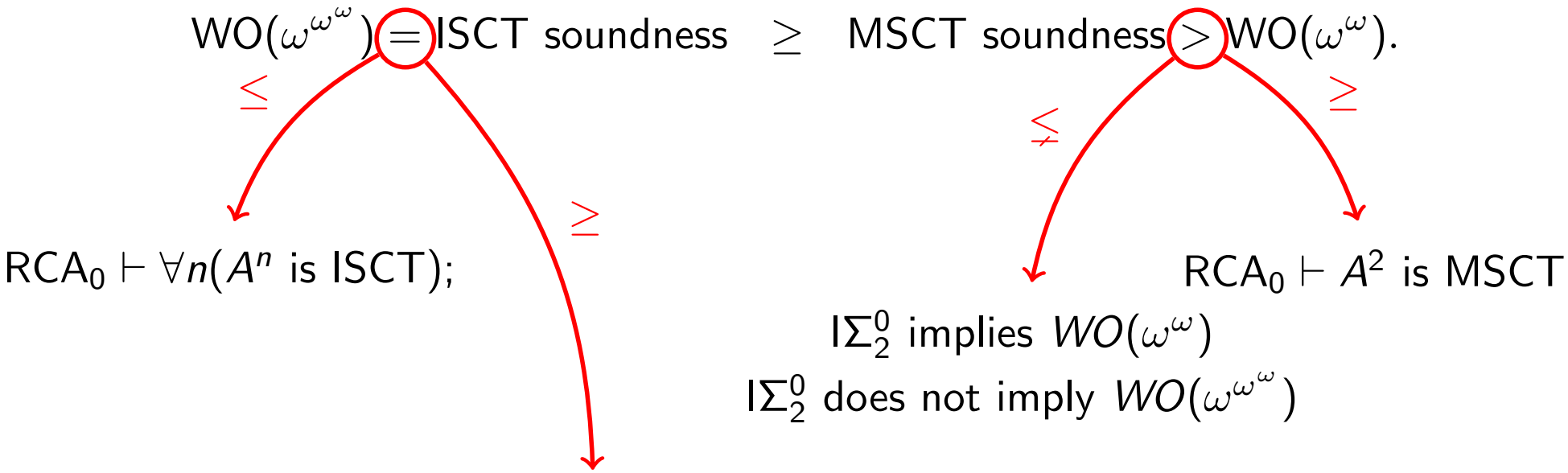
- Let $\mathcal{G}$ be a safe description of a program P and suppose that P does not terminate on some input.
- Then there exists an infinite sequence of state transitions.

existence of an infinite branch in the so called activation tree!

- Consider the corresponding multipath in $\mathcal{G}$.
- As $\mathcal{G}$ is MSCT, there exists an infinite descent.
- Safety provides a contradiction!

Where can we prove soundness?

Over RCA_0:

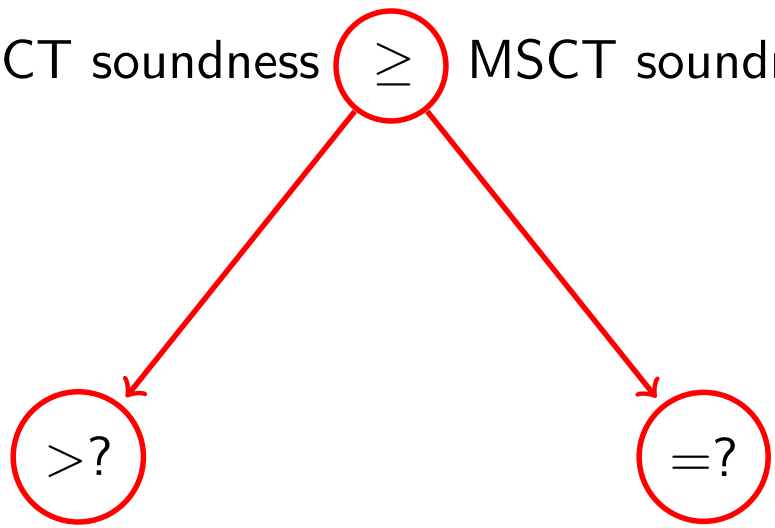


Given an infinite reduction sequence we assign decreasing ordinals below ω^{ω^ω}

Where can we prove soundness?

Over RCA_0:

$$\mathrm{WO}(\omega^{\omega^\omega}) = \text{ISCT soundness} \geq \text{MSCT soundness} > \mathrm{WO}(\omega^\omega).$$



More about Triang

Let denote with $[n]^2$ the complete graph on n-nodes and with $[n]^3$ the complete 3-hypergraph with n-nodes. Let $([n]^2\text{-}\mathrm{RT}_k)$ be:

For any coloring $c : [\mathbb{N}]^2 \to k$ there exist $i \in k$ and $t \in \mathbb{N}$ such that $c[[\{t, m_0, \ldots, m_{n-1}\}]^2] = \{i\}$ for infinitely many tuple $\{m_0, \ldots, m_{n-1}\}$.

- (Over RCA_0) For every n and k, $[n]^2\text{-}\mathrm{RT}_k$ is equivalent to $\mathrm{I}\Sigma^0_2$.
- ACA_0 proves $[n]^3\text{-}\mathrm{RT}_k$ for every n and k.

Which is the strength of $[n]^3\text{-}\mathrm{RT}_k$?

Thank you!

Triang implies the SCT criterion.

- Let $\mathcal{M}_\pi = G_0, \ldots, G_n, \ldots$.
- Define $c(i,j) = G_i; \ldots; G_{j-1}$.
- By $\mathrm{Triang}_{|\mathsf{cl}(\mathcal{G})|}$ to c, there exist t and $G \in \mathsf{cl}(\mathcal{G})$ such that

$$\forall n \exists m, l (n < m < l \wedge t < m \wedge c(t,m) = c(t,l) = c(m,l) = G).$$

- G is idempotent, indeed $G; G = c(t,m); c(m,l) = c(t,l) = G$.
- By hypothesis, we have that there exists $x \overset{\downarrow}{\rightarrow} x \in G$.
- We prove that there exists an infinite descent starting from x in G_t.

SCT criterion implies SPP - Case $k = 2$

- $\mathcal{G} = \{G_0, G_1, G_2\}$.
- For $i < 3$, the graph G_i has only one strict arc $z_i \overset{\downarrow}{\to} z_i$ and non-strict arcs $z_j \overset{\Downarrow}{\to} z_j$ for $j > i$.
- Define:

$$g(x) = \begin{cases} 0 & \text{if } c(x) = 0 \wedge c(x+1) = 0 \\ 1 & \text{if } c(x) = 1 \wedge c(x+1) = 1 \\ 2 & \text{otherwise} \end{cases}$$

- Consider the multipath $M = G_{g(0)}, G_{g(1)}, \ldots$ By the SCT criterion, it has an infinite descent.
- There exists a parameter z_i such that $z_i \overset{\downarrow}{\to} z_i \in G_{g(x)}$ $(g(x) = i)$ for infinitely many x.
- If $i < 2$, it means that from some point on $c(x) = i$ and so $I = \{i\}$.
- If $i = 2$ the color changes infinitely many times and so $I = \{0, 1\}$.

$\mathrm{I}\Sigma^0_2$ implies Triang

- (Slaman and Yokoyama) RT^2 is Π^1_1-conservative over $\mathrm{B}\Sigma^0_3$.
- A statement is $\widetilde{\Pi}^0_4$ if it is of the form $\forall X \varphi(X)$ and $\varphi(X) \in \Pi^0_4$.
- (e.g. Hajek and Pudlák) $\mathrm{B}\Sigma^0_3$ is $\widetilde{\Pi}^0_4$-conservative over $\mathrm{I}\Sigma^0_2$.
- Triang is $\widetilde{\Pi}^0_4$.

Where $\mathrm{B}\Sigma^0_3$ is the the bounding principle for Σ^0_3-formulas: for every σ^0_3 formula φ:

$$\forall m[(\forall i < m \exists u \varphi(i,u)) \implies \exists v(\forall i < m \exists u < v \varphi(i,u))]$$