Definable versions of algebraic equivalents of CH

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Törnquist and Weiss idea

In 2012 Törnquist and Weiss studied many Σ^1_2 definable version of some statements equivalent to CH ($2^{\aleph_0} = \aleph_1$).

CH $\iff$ there exist some **objects** such that **something happens**.

They proved that these Σ^1_2 counterparts become equivalent to the statement "all reals are constructible".

$\mathbb{R} \subseteq L$ $\iff$ there exist some Σ^1_2 **objects** such that **something happens**.

From "CH implies S" to "$\mathbb{R} \subseteq L$ implies the Σ^1_2 version of S"

A Δ^1_2 well-ordering $\prec$ is **strong** if it has length ω_1 and if for any $P \subseteq \mathbb{R} \times \mathbb{R}$ which is Σ^1_2,

$$\forall z \prec y \; P(x, z)$$

is Σ^1_2 as well.

Theorem (Addison 1959)

If $\mathbb{R} \subseteq L$ then there exists a Δ^1_2 strong well-ordering of the reals.

From "S implies CH" to "the Σ^1_2 version of S implies $\mathbb{R} \subseteq L$"

Theorem(Mansfield and Solovay 1970)

Let A be a $\Sigma^1_2(a)$ set. Then either $A \subseteq L[a]$, or else A contains a perfect set. In particular, if a Σ^1_2 set contains a non-constructible real then it is uncountable.

Lemma (Törnquist and Weiss 2012)

1. If there exists a non-constructible real, there exists a non-constructible real $x \in V$ such that $\aleph_1^{L[x]} = \aleph_1^L$.
2. Let $a \in L$ and A be a $\Sigma^1_2(a)$ definable set. Then if A is uncountable, $A \cap L$ is uncountable in L.

Törnquist and Weiss results

Theorem (Sierpinski 1965)

CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there are Σ^1_2 sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

Törnquist and Weiss results

Theorem (Sierpinski 1965)

CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ holds iff there are Σ^1_2 sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

Törnquist and Weiss results

Theorem (Komjáth and Totik 2006)

$\neg$CH implies that for any coloring $g : \mathbb{R} \to \omega$ there are four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \not\subseteq L$ iff for any Σ^1_2 coloring $g : \mathbb{R} \to \omega$ there are four distinct $x, y, z, w \in \mathbb{R}$ of the same color such that

$$x + y = z + w.$$

What does it happen if we have infinitely many objects?

Theorem (Erdős and Kakutani 1943)

CH holds iff the set of all real numbers can be decomposed into a countable number of rationally independent subsets.

A set $X \subseteq \mathbb{R}$ is rationally independent if either $X = \{0\}$ or for any $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$ and for any $q_0, \ldots, q_{n-1} \in \mathbb{Q} \setminus \{0\}$ we have

$$\sum_{i=0}^{n-1} q_i x_i \neq 0.$$

What does it happen if we have infinitely many objects?

Theorem (Erdős and Kakutani 1943)

CH holds iff the set of all real numbers can be decomposed into a countable number of rationally independent subsets.

Proposition

$\mathbb{R} \subseteq L$ iff the set of all real numbers can be decomposed into a countable number of uniformly Σ^1_2 definable rationally independent subsets.

What does it happen if we have infinitely many objects?

Theorem (Zoli 2006)

CH holds iff the set of all transcendental reals is union of countably many transcendence bases for $\mathbb{R}$.

Proposition

$\mathbb{R} \subseteq L$ iff the set of all transcendental reals is union of countably many uniformly Σ^1_2 definable algebraically independent sets.

Thanks to the Σ^1_2 versions of Schmerl's results. . .

Theorem (Schmerl 1999)

If $\neg$CH holds then every avoidable polynomial is 2-avoidable.

Proposition

If $\mathbb{R} \not\subseteq L$ then every Σ^1_2-avoidable polynomial is $(2, \Sigma^1_2)$-avoidable.

Thanks to the Σ^1_2 versions of Schmerl's results...

Theorem (Schmerl 1999)

If CH holds then every 1-avoidable polynomial is avoidable.

Proposition

If $\mathbb{R} \subseteq L$ then every $(1, \Sigma^1_2)$-avoidable polynomial is Σ^1_2-avoidable.

Erdős and Komjáth equivalence!

Theorem (Erdős and Komjáth 1990)

CH holds if and only if the plane can be colored with countably many colors with no monochromatic right-angled triangle.

Proposition

$\mathbb{R} \subseteq L$ if and only if there exists a Σ^1_2 coloring of the plane with countably many colors with no monochromatic right-angled triangle.

Erdős and Komjáth equivalence!

Theorem (Erdős and Komjáth 1990)

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Proposition

$\mathbb{R} \subseteq L$ if and only if there exists a Σ^1_2 coloring of the plane with countably many colors with no monochromatic right-angled triangle.

Thank you!