When the reals form a proper class

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$\mathbf{KP}^c$

- Let $\mathcal{L}^c$ be the extension of $\mathcal{L}$ with countably many class variables.
- The atomic formulas comprise the ones of $\mathcal{L}$ and all expression of the form "$a \in C$".
- An $\mathcal{L}^c$ formula is elementary if it contains no class quantifiers.
- Δ_n^c, Σ_n^c and Π_n^c are defined as usual, but permitting subformulas of the form "$a \in C$".

KP^c

The theory KP^c is formulated in $\mathcal{L}^c$ and consists of the following axioms:

- extensionality, pair, union, infinity,
- Δ_0^c-Separation: i.e, for every Δ_0^c formula φ in which x is not free and any set a,
$$\exists x(x = \{y \in a : \varphi[y]\})$$
- Δ_0^c-Collection: i.e, for every Δ_0^c formula φ and any set a,
$$\forall x \in a \exists y \varphi[x, y] \to \exists b \forall x \in a \exists y \in b \varphi[x, y]$$
- Δ_1^c-Comprehension: i.e, for every Σ_1^c formula φ and every Π_1^c formula ψ,
$$\forall x(\varphi[x] \leftrightarrow \psi[x]) \to \exists X \forall x(x \in X \leftrightarrow \varphi[x])$$
- Elementary $\in$-induction: i.e, for every elementary formula φ,
$$\forall x((\forall y \in x \varphi[y]) \to \varphi[x]) \to \forall x \varphi[x]$$

Motivations: operators!

- We call a class an operator if all its elements are ordered pairs and it is right-unique (i.e. functional).
- We use F to denote operators.
- Given an operator F and a set a we write $\mathrm{Mon}[F, a]$ for:

$$\forall x(F(x) \subseteq a) \wedge \forall x, y(x \subseteq y \rightarrow F(x) \subseteq F(y)).$$

Least fixed point statements

FP

$$\mathrm{Mon}[F, a] \to \exists x(F(x) = x)$$

LFP

$$\mathrm{Mon}[F, a] \to \exists x(F(x) = x \wedge \forall y(F(y) = y \to x \subseteq y))$$

Separation

Σ_1^c-separation

For every Σ_1^c formula φ in which x is not free and any set a,

$$\exists x(x = \{y \in a : \varphi[y]\}).$$

SBS ($\sim \Pi_1^{\mathcal{P}}(\Delta_1^c)$-Sep)

For every Δ_1^c formula φ and sets a and b,

$$\exists z(z = \{x \in a : \exists y \subseteq b(\varphi[x, y])\})$$

Fixed point principles in $\mathsf{KP}^c + (V{=}L)$

$$
\begin{tikzcd}
 & \Sigma_1^c\text{-Sep} \arrow[dl] \arrow[dr] & \\
\mathsf{MI} \arrow[d] \arrow[drr] & & \mathsf{SBS} \arrow[dll] \arrow[d] \\
\mathsf{BPI} \arrow[uur, bend left, "(V{=}L)"] & & \mathsf{LFP} \arrow[d] \\
 & & \mathsf{FP} \arrow[uu, bend right, "(V{=}L)"']
\end{tikzcd}
$$

If we add to our theory the Axiom of Limitation of Size:

- we have a global well-ordering of V,
- all our principles are equivalent,
- But... I am not able to prove the consistency of:
$$\mathsf{KP}^c + \mathsf{FP} + \text{Limitation of size},$$
from the consistency of $\mathsf{KP}^c + \mathsf{FP}$.

What does it happen if we consider something weaker than a bijection?

Injections from ordinals to reals

Proposition

Assume that there are no injections from Ord to $\mathcal{P}(\omega)$. Then MI hold!

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Question

And if there is an injection from Ord to $\mathcal{P}(\omega)$?

Injections from reals to ordinals

Proposition

Assume that there is an injection from $\mathcal{P}(\omega)$ to Ord. Then BPI implies MI.

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Assume that there is an injection from $\mathcal{P}(\omega)$ to Ord. Then BPI implies MI.

Question

Assume that there are no injections from $\mathcal{P}(\omega)$ to Ord... BPI holds.

Surjections from ordinals to reals

Proposition

Assume that there is a surjection from Ord to $\mathcal{P}(\omega)$. Then there exists a strong well ordering of $\mathcal{P}(\omega)$.

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Assume that there is a surjection from Ord to $\mathcal{P}(\omega)$. Then there exists a strong well ordering of $\mathcal{P}(\omega)$.

Question

Which is the strength of the statement: "For every class C, there exists either an injection from C to the ordinals or a surjection from the ordinals to C"?

Cofinal maps from reals to ordinals

Theorem

Assume that there exists a cofinal map $F : \mathcal{P}(\omega) \to \mathrm{Ord}$. Then SBS implies Σ^c_1-Separation for ordinals.

- Given φ we want to show that $\{x \in \omega : \exists \alpha \varphi[\alpha, x]\}$ is a set.
- By using F:

$$\exists \alpha \varphi[x, \alpha] \iff \exists y \subseteq \omega (\exists \alpha < F(y)(\varphi[x, \alpha])).$$

- The formula "$\exists \alpha < F(y)(\varphi[x, \alpha])$" is Δ^c.
- By applying SBS we get the thesis.

Cofinal maps from reals to ordinals

Theorem

Assume that there are no cofinal maps from the reals to the ordinals. Then Π_1-Reduction for ordinals holds.

Π_1-Reduction for ordinals

Let φ and ψ be two Δ_0 formulas such that

$$\forall x \in \omega (\exists \alpha \varphi[x, \alpha] \implies \forall \alpha \psi[x, \alpha]).$$

there exists a set z such that

$$\{x \in \omega : \exists \alpha \varphi[x, \alpha]\} \subseteq z \subseteq \{x \in \omega : \forall \alpha \psi[x, \alpha]\}.$$

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Thank you!