Proving termination with transition invariants of height ω

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Termination Theorem by Podelski and Rybalchenko

- Transition invariants are used by Podelski and Rybalchenko to prove the termination of a program.
- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$.
- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem (Termination Theorem)

The program P is terminating iff there exists a disjunctively well-founded transition invariant for P.

The proof by Podelski and Rybalchenko requires Ramsey Theorem

If you have ω people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem (Ramsey for pairs)

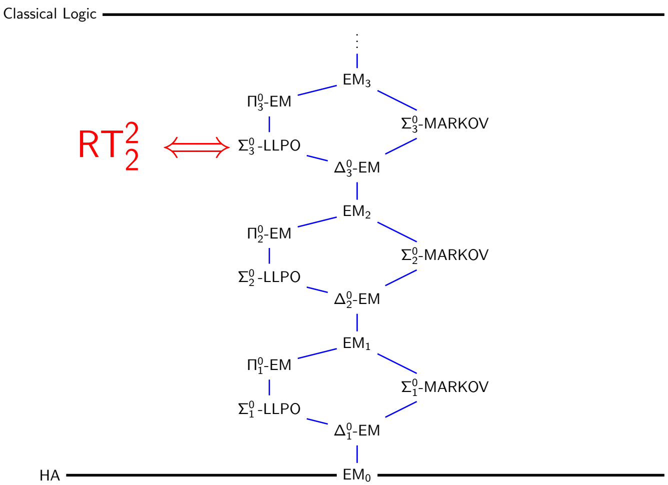
Assume I is a set having some injective enumeration $\sigma = x_0, x_1, \ldots, x_i, \ldots$. Assume $S_1, \ldots, S_n$ are binary relations on I which are a partition of $\{(x_i, x_j) \in I \times I : j < i\}$, that is:

1. $S_1 \cup \cdots \cup S_n = \{(x_i, x_j) \in I \times I : j < i\}$
2. *for all* $1 \leq k < h \leq n$*:* $S_k \cap S_h = \emptyset$.

Then for some $k \in [1, n]$ there exists some infinite $X \subseteq \mathbb{N}$ such that:

$$\forall i, j \in X.(j < i \implies x_i S_k x_j).$$

How many steps before the program P terminates? Hard to say, because Ramsey Theorem is a purely classical result.



We provided a new intuitionistic version of Ramsey Theorem, which we called H-closure Theorem, and a new intuitionistic proof of the Termination Theorem.

Why did we care?

In order to use our proof to extract some effective bounds for the Termination Theorem.

H-closure Theorem

In order to analyse the bounds implicit in the Termination Theorem we replaced Ramsey with some intuitionistic result.

Let R be a binary relation on I.

- $H(R)$ is the set of the R-decreasing transitive finite sequences on I:

$$\langle x_1, \ldots, x_n \rangle \in H(R) \iff \forall i, j \in [1, n](i < j \implies x_j R x_i).$$

- R is H-well-founded if $H(R)$ is $\succ$-well-founded.

Theorem (H-closure)

The H-well-founded relations are closed under finite unions:

$$(R_1, \ldots, R_n \text{ } H\text{-well-founded}) \implies ((R_1 \cup \cdots \cup R_n) \text{ } H\text{-well-founded}).$$

Our goal

Theorem

A function has at least one implementation which has a disjunctively well-founded transition invariant whose relations are primitive recursive and have height ω if and only if it is primitive recursive.

Our goal

Theorem

Assume that P is such that $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) = R_1 \cup \cdots \cup R_k$, where

1. *the complement of the set of the final states of R ($S \setminus F$) is a primitive recursive set;*
2. *R is the graph of a primitive recursive function $t : S \setminus F \to S$: i.e.*

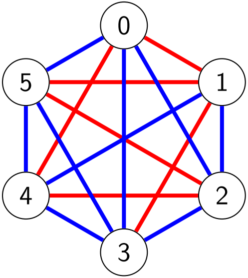
$$R = \{(x, t(x)) : x \in S \setminus F\} .$$

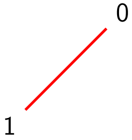
3. *$R_1, \ldots, R_k$ are primitive recursive relations and have height ω.*

Then there exists a primitive recursive $g : S \to \mathbb{N}$ such that $t^{g(s)}(s) = t^{g(s)+1}(s)$.

Erdős' trees

Assume given a sequence $\langle 0, \ldots, 5 \rangle$ such that the coloring between its elements is as follows.





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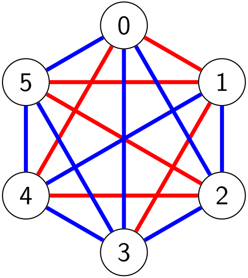

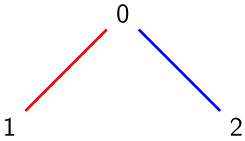

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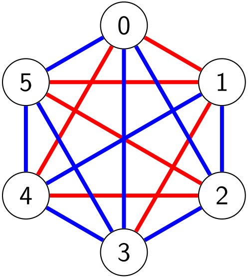

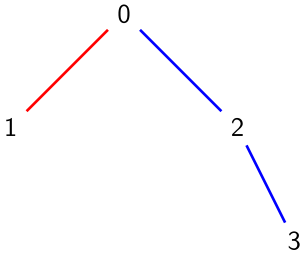

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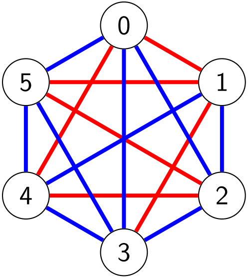

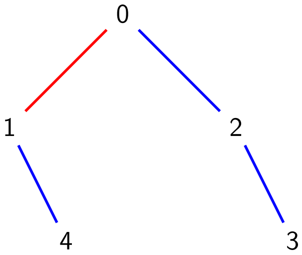

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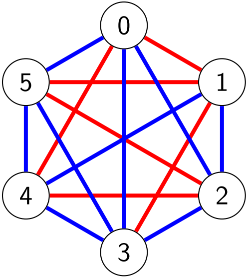

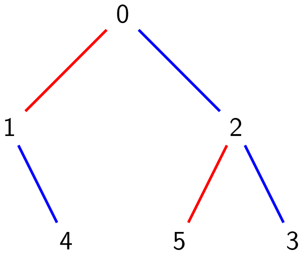

Sketch of the proof

- There exists a primitive recursive function from the set of k-branching Erdős' trees to ω^k, which we called h_k, which "preserves" the one step expansion between trees:

$$T' \succ_E T \implies h_k(T') < h_k(T).$$

- There exists a primitive recursive function E from $H(R_0 \cup \cdots \cup R_{k-1})$ to the set of the Erdős' trees (as we did in the previous example), which "preserves" the one step expansion between sequences:

$$L' \succ L \implies E(L') \succ_E E(L).$$

So given an R-chain α we want to prove that it is finite.

- Define

$$\begin{aligned} \theta : \mathbb{N} &\to H(R_0 \cup \ldots R_{k-1}) \\ m &\mapsto \langle s_0, t(s_0), \ldots, t^m(s_0) \rangle \end{aligned}$$

By definition we have:

$$m' > m \implies \theta(m') \succ \theta(m).$$

- Let $\phi : \mathbb{N} \to \mathbb{N}^k$ be such that $\phi(m) = h_k(E(\theta))$. It is primitive recursive!

Lemma

For each $\sigma : \mathbb{N} \to \mathbb{N}^k$ *primitive recursive, there exists* $g : \mathbb{N} \to \mathbb{N}$ *primitive recursive such that*

$$\forall n \exists m \in [n, g(n)](\sigma(m) \preccurlyeq_k \sigma(m+1)).$$

By applying this lemma to ϕ we obtain the primitive recursive bound we were looking for!

What are we doing now?

Since by Termination Theorem Podelski, Rybalchenko and Cook produced an algorithm, we hope that H-closure Theorem could be useful in order to find bounds also for the algorithm.

Conjecture

A function has at least one implementation in Podelski-Rybalchenko language which the Terminator Algorithm may catch terminating if and only if the function is primitive recursive.

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