Regular Sets of Trees and Probability

Matteo Mio
CNRS & ENS–Lyon

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- Presburger arithmetic $\mathsf{FO}(\mathbb{N}, <, +)$,
- Linear arithmetic $\mathsf{FO}(\mathbb{R}, <, +)$,
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and

- (temporal) logics in computer science:
  - MSO(words), LTL,
  - MSO(trees), CTL, CTL*, μ-calculus, ...

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- Models: Labeled Trees

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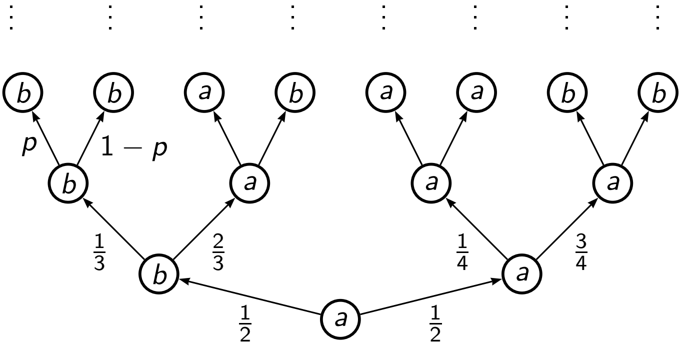
Probabilistic Temporal Logics

- Example: *Probabilistic Computation Tree Logic* (pCTL)
- Models: Labeled Markov Chains (= trees with probabilities)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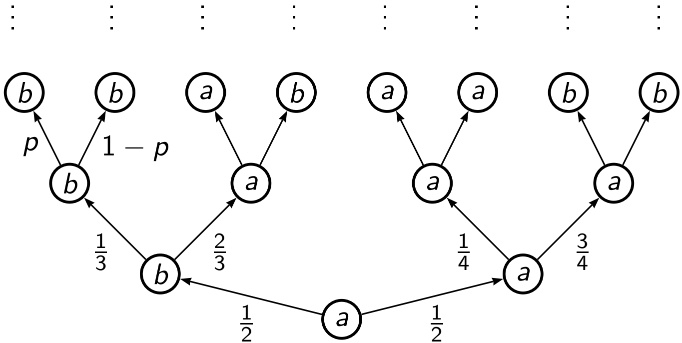
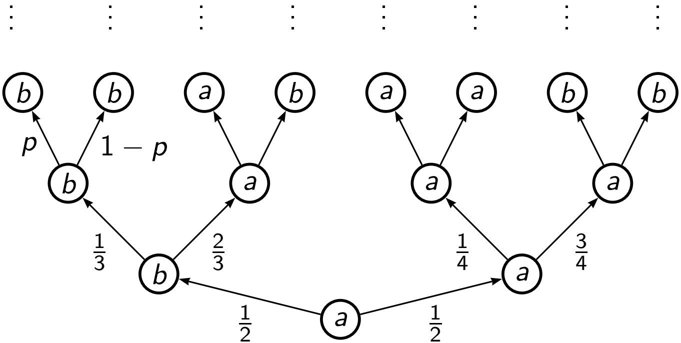


A pCTL formula: $\mu\big(\pi \mid \pi \text{ has infinitely many } a\big) \geq \frac{1}{3}$



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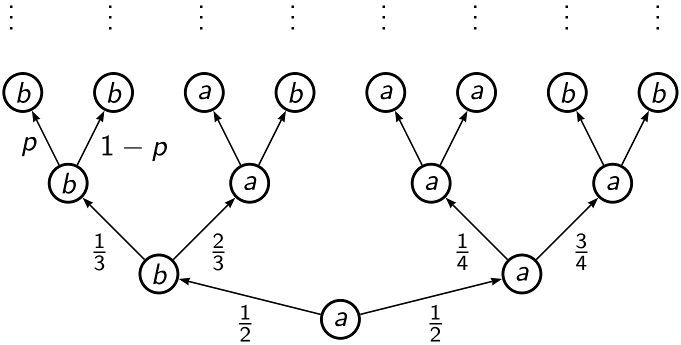

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Open Problem (Lehmann–Shelah, 82). Given a formula ϕ

$$\exists M.(M \models \phi)?$$

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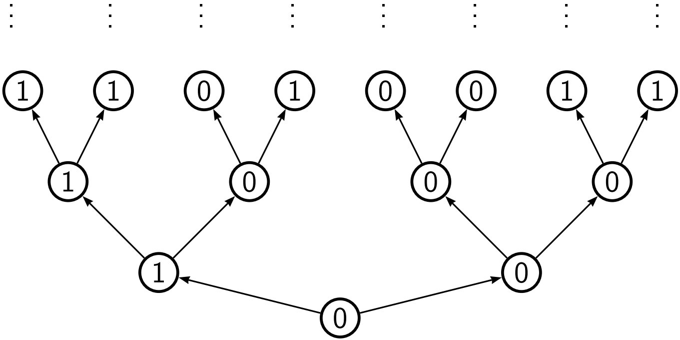

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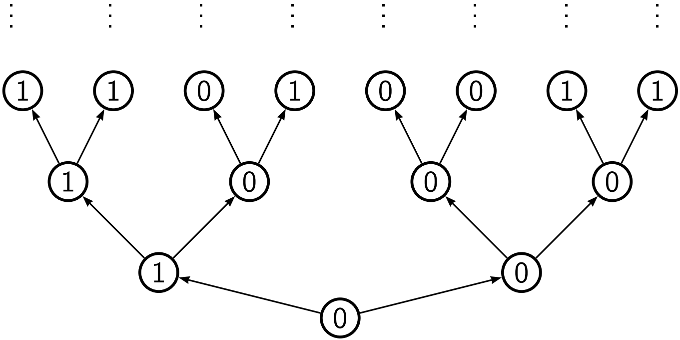

Definition: A set $L \subseteq \mathcal{T}_{\{0,1\}}$ is *regular* if it is definable by a S2S formula $\phi(X)$.

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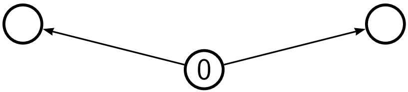

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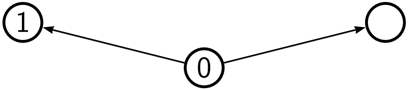

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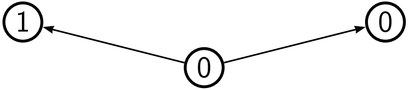

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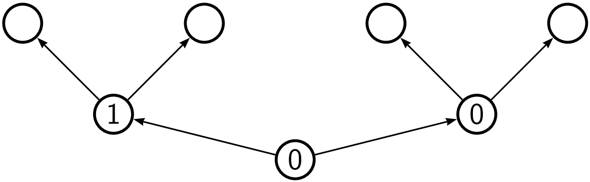

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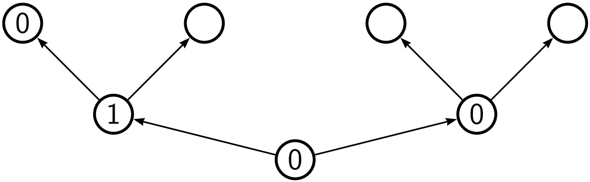

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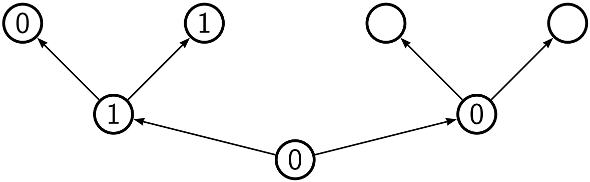

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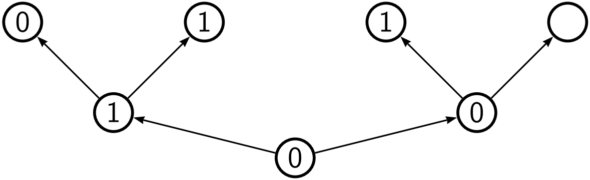

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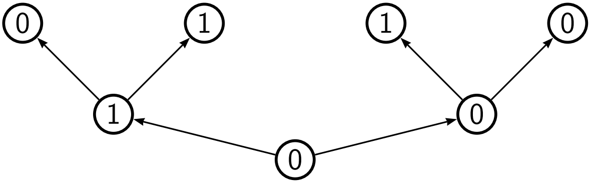

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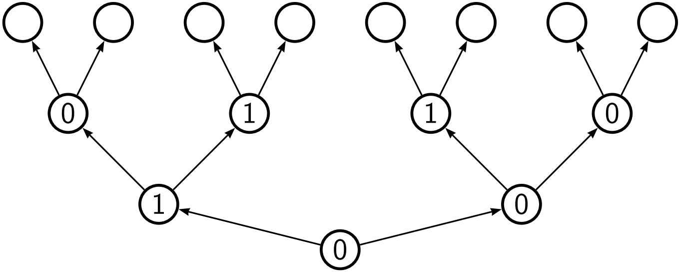

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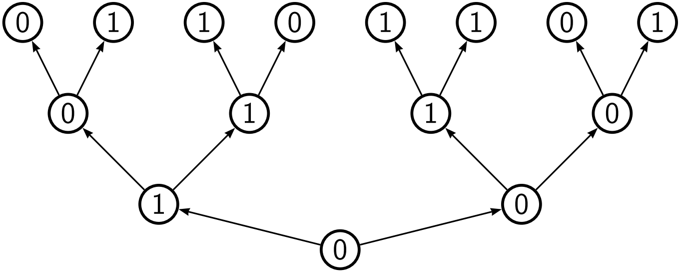

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Question 0: are all regular sets μ-measurable?

Measurability of Regular Sets

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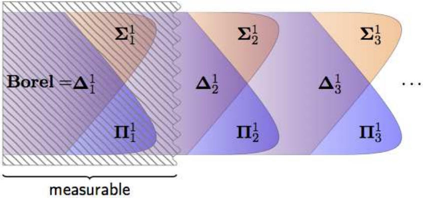

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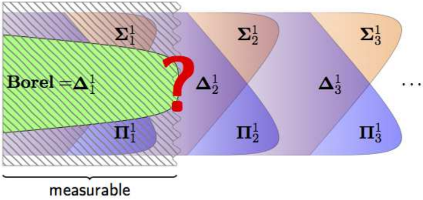

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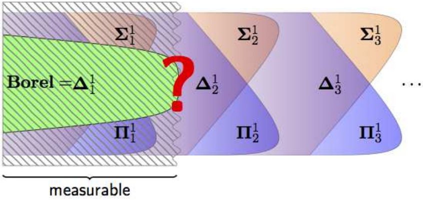

Spoiler: answer is yes.

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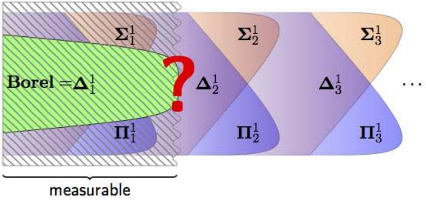

Spoiler: answer is yes.

- Using a rather advanced theorem (proved using *forcing*) from set-theory.

J. Fenstad and D. Normann,
On absolutely measurable sets, Fundamenta Mathematicae, 1974.

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Kolmogorov's σ-algebra of $\mathcal{R}$-sets: $\sigma\Big(\texttt{Open}, \{\mathcal{R}^n\}_n, \neg\Big)$

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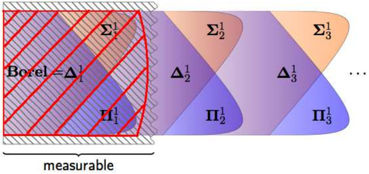

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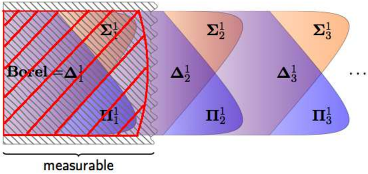
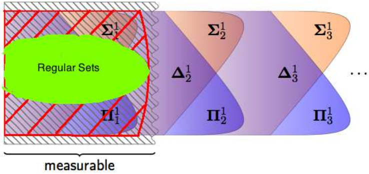


Theorem: (Gogacz, Michalewski, Mio, Skrzypczak, 2017)



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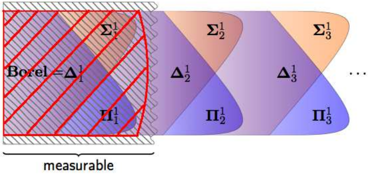

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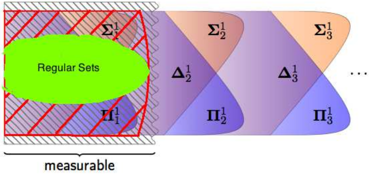

- *Game languages* $W_{i,k}$ are complete for the finite levels of the hierarchy of $\mathcal{R}$-sets.

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Open problem

A partial solution.

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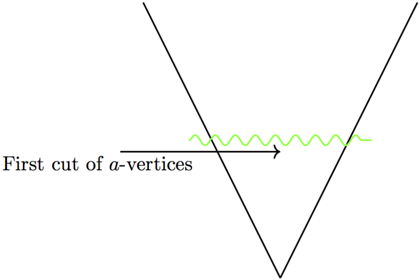
Our algorithm is quite involved:

1. Reduction to a systems of nested (co)inductive polynomial equations,
2. Solve these equations using Tarski's decision procedure for $FO(\mathbb{R}, +, \times, 0, 1)$.

$L_1 \subseteq \mathcal{T}_{\{a,b,c\}}$



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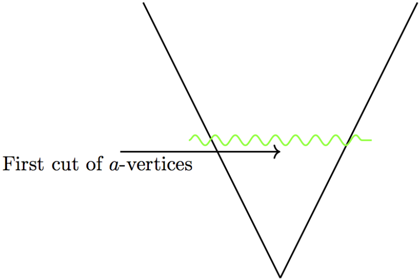
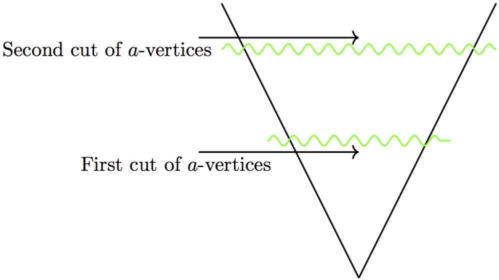


Algorithm calculates: $\mu(L_1) = \frac{1}{2}$

$L_2 \subseteq \mathcal{T}_{\{a,b,c\}}$



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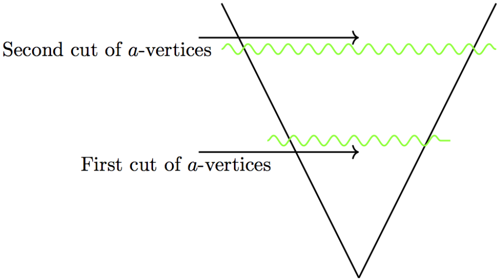


Algorithm calculates: $\mu(L_2) = \frac{1}{4}(3 - \sqrt{7}) \approx 0.088$

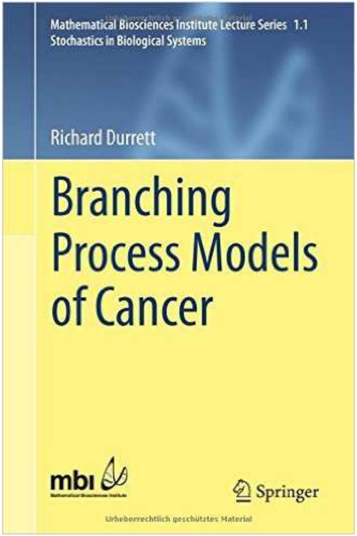
One application

Model Checking of Markov Branching Processes

$R \overset{0.1}{\rightsquigarrow} D$
$R \overset{0.89}{\rightsquigarrow} (R, R)$
$R \overset{0.01}{\rightsquigarrow} M$

$M \overset{0.9}{\rightsquigarrow} D$
$M \overset{0.1}{\rightsquigarrow} (R, M, M, M, M)$

$D \overset{1}{\rightsquigarrow} D$


Mathematical Biosciences Institute Lecture Series 1.1
Stochastics in Biological Systems
Richard Durrett
Branching Process Models of Cancer
mbi
Springer
Urheberrechtlich geschütztes Material

Other curiosities

Lemma (Measure $\neq$ Baire Category)

There are regular sets L such that:

$$\mu(L) = 0 \qquad \textit{and} \qquad L \textit{ is comeager (second category)}$$

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Lemma (A Zero-One law)

The language $W_{i,k} \subseteq \mathcal{T}_\Sigma$ *with* $\Sigma = \{\forall, \exists\} \times \{i, \ldots, k\}$

$$\mu(W_{i,k}) = 1 \ \textit{ if } k \textit{ is even} \qquad \mu(W_{i,k}) = 0 \ \textit{ if } k \textit{ is odd.}$$

Some Questions (possibly easy)

Theorem (Regularity)

For every measurable $A \subseteq \mathcal{T}_\Sigma$, *there is a* G_δ *set* $B \supseteq A$ *such that* $\mu(A) = \mu(B)$.

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Theorem (Regularity)

For every measurable $A \subseteq \mathcal{T}_\Sigma$*, there is a* G_δ *set* $B \supseteq A$ *such that* $\mu(A) = \mu(B)$*.*

Question: Given $A \subseteq \mathcal{T}_\Sigma$ regular, can we find B regular?

Theorem

Let $E \subseteq [0,1]$ *a Borel set. Then there exists a basic interval* (a, b) *such that*

$$\mu\big(E \ \cap \ (a,b)\big) \in \{0, b-a\}$$

(i.e., E *has full or null measure in* (a, b)*).*

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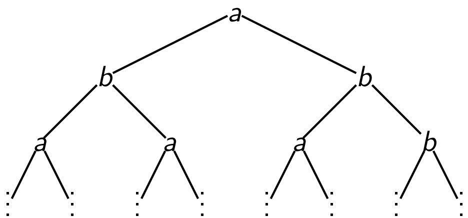
Question: Is the theorem true for $E \subseteq \mathcal{T}_\Sigma$ regular?

Generalised Quantifiers



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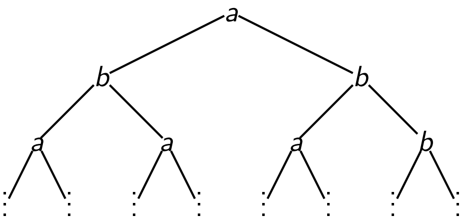

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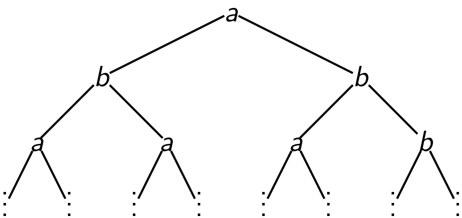

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Properties we wish to express in probabilistic logics:

- $\big\{X \mid \text{“}X \text{ is a branch”} \wedge \phi(X)\big\}$ has probability $= 1$.

Definition (Syntax of MSO+$\forall^{=1}_{\pi}$)

$$\phi ::= \neg\phi \mid \phi_1 \vee \phi_2 \mid \forall x.\phi \mid \forall X.\phi \mid x \in X$$

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Question: Is MSO+$\forall^{=1}_{\pi}$ decidable?

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$$\{X \mid X \text{ is a branch and } \phi(X) \text{ holds}\}$$

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$$\forall^{=1} X.\ \phi(X) \text{ holds}$$

$$\Leftrightarrow$$

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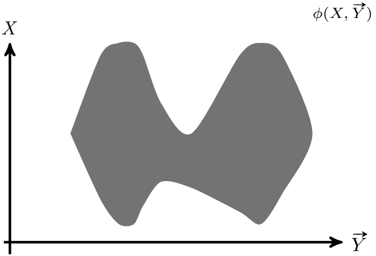
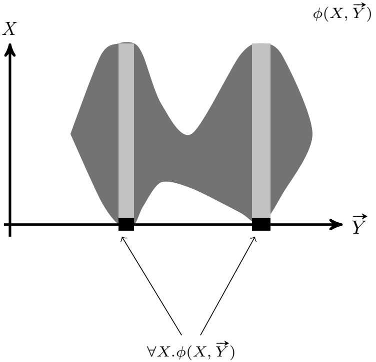
A formula $\phi(X, \vec{Y})$ defines a set in the product space:

$$\mathcal{T}_\Sigma \times \mathcal{T}_\Sigma^n$$

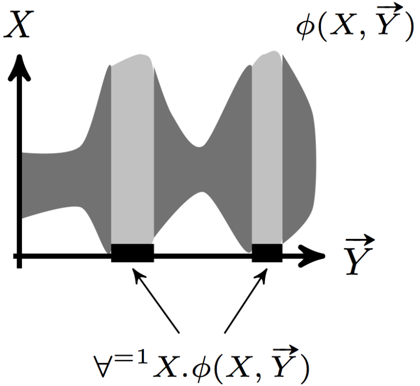


Interpretation of the $\forall$ quantifier:



Takes *full* sections.

Interpretation of the $\forall^{=1}$ quantifier:



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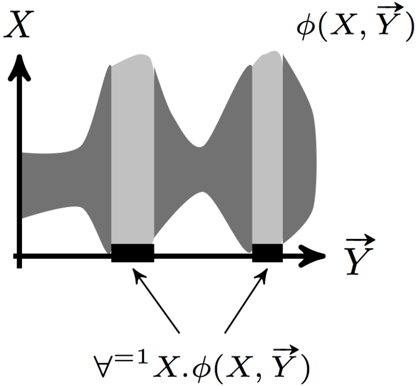

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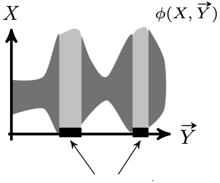
Quantifier $\forall^{=1} X.\phi(X)$ first studied by H. Friedman in 1979.

Interpretation of the $\forall^*$ quantifier:



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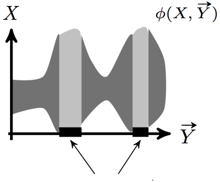

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Theorem (Friedman, on Borel structures)

The theories of FO $+ \forall^{=1}$ and FO $+ \forall^*$ coincide.

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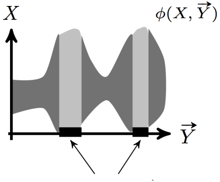

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Theorem (Friedman, on Borel structures)

The theories of FO $+ \forall^{=1}$ *and FO* $+ \forall^*$ *coincide.*

Theorem (Staiger, 97)

A regular $L \subseteq \Sigma^\omega$ *is comeager iff it has measure* 1.

Some Results

Mio, Skrzypczak, Michalewski:

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in LMCS, vol 14(2), 2018.

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$\mathsf{MSO} + \forall^{=1} \supsetneq \mathsf{MSO}$ *and undecidable.*

Theorem (on trees)

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- *(Bojanczyk)* $\mathsf{WMSO} + \forall_{\pi}^{=1}$ *is decidable.*

My Conclusion:

Tree–Automata Theory + Probability $\neq$ Easy

THANKS